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SUR LES OPÉRATEURS DE SCHRÖDINGER ALÉATOIRES DANS LE CONTINU

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S'il faut aller au cimetière,
Je prendrai le chemin le plus long,
Je ferai la tombe buissonnière,
Je quitterai la vie à reculons.
Tant pis si les croque-morts me grondent,
Tant pis s'ils me croient fou à lier.
Je veux partir pour l'autre monde,
Par le chemin des écoliers.

Georges Brassens

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« Par le chemin des écoliers », ainsi chante Brassens. L'expression fait référence à « la propension des élèves à parfois traîner des pieds et à se laisser tenter par les chemins de traverse en allant en cours ». Pour moi, cela signifie aussi les détours que l'on a fait sur la route vers les connaissances : l'important, c'est d'apprendre sur le chemin, et de prendre du plaisir. En fin de compte, nous, les mathématiciens, sommes tous dans un sens des « écoliers », n'est-ce pas ?

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> 因爲需要感謝的人太多了，就感謝天罷。

Resumé

Cette thèse porte sur les opérateurs de Schrödinger aléatoires dans un cadre continu, en particulier ceux avec un potentiel de bruit blanc gaussien. La définition de ces opérateurs différentiels est généralement non triviale et nécessite la renormalisation dans les dimensions $d \geq 2$. Nous présentons d'abord un cadre général pour traduire le problème de construction de l'opérateur en EDP stochastiques. Cette approche nous permet de définir l'opérateur en question, d'établir son auto-adjonction et d'étudier son spectre.

Par la suite, nous passons à l'étude de l'Hamiltonien d'Anderson continu dans deux configurations spatiales distinctes : d'abord dans une boîte bornée de longueur latérale L avec une condition de bord de Dirichlet nulle pour les dimensions $d \leq 3$, et ensuite dans l'espace Euclidien $\mathbb{R}^d$, pour $d \in \{2, 3\}$. Dans le premier cas, l'opérateur admet des valeurs propres $\lambda_{n,L}$, pour lesquelles nous identifions l'asymptotique presque sûre lorsque $L \to \infty$. Cet asymptotique est conforme aux résultats antérieurs dans la littérature pour les dimensions 1 et 2, tandis que notre résultat en dimension 3 est nouveau. Dans le second cas, nous proposons une nouvelle technique de construction en utilisant la théorie des solutions de l'équation parabolique associée, ce qui permet de prouver l'auto-adjonction et de montrer que le spectre est presque sûrement égal à $\mathbb{R}$. Cette approche confirme le résultat récemment établi en dimension 2 dans la littérature, cependant notre construction semble plus élémentaire ; pour la dimension 3, notre résultat est nouveau.

Enfin, nous présentons un projet en cours qui aborde le cas où un champ magnétique uniforme est appliqué au système : cela conduit à l'étude de l'Hamiltonien de Landau perturbé par le potentiel de bruit blanc. Notre objectif est de définir l'opérateur dans l'espace $\mathbb{R}^2$ sans recourir à une théorie de renormalisation sophistiquée. Cependant, la non-bornitude du bruit blanc sur $\mathbb{R}^2$ pose des défis techniques supplémentaires. Pour surmonter cela, l'utilisation du théorème de Faris-Lavine est discutée.

Abstract

This thesis studies the random Schrödinger operators in continuous setting, particularly those with Gaussian white noise potential. The definition of such differential operators is generally non-trivial and necessitates renormalization in dimensions $d \geq 2$. We first present a general framework to translate the problem of operator construction into stochastic PDEs. This approach enables us to define the operator at stake and establishes its self-adjointness, as well as to investigate its spectrum.

Subsequently, we proceed to study the continuous Anderson Hamiltonian under two distinct spatial settings: first on a bounded box with side length L with zero Dirichlet boundary condition for dimensions $d \leq 3$, and second on the full Euclidean space $\mathbb{R}^d$, for $d \in \{2, 3\}$. In the former case, the operator admits eigenvalues $\lambda_{n,L}$, for which we identify the almost sure asymptotic as $L \to \infty$. This asymptotic aligns with previous findings in the literature for dimension 1 and 2, while our result in dimension 3 is new. In the latter case, we propose a new construction technique employing the solution theory to the associated parabolic equation which allows to prove self-adjointness and show that the spectrum equals to $\mathbb{R}$ almost surely. This approach reconfirms the recently established result in dimension 2, but our construction seems to be more elementary; for dimension 3, our result is new.

Lastly, we present an ongoing project addressing the case where a uniform magnetic field is applied to the system: this leads to the study of Landau Hamiltonian perturbed by the white noise potential. Our objective is to define the operator on full space $\mathbb{R}^2$ without resorting to sophisticated renormalization theory. However, the unboundedness of white noise on $\mathbb{R}^2$ poses additional technical challenges. To overcome this, the usage of Faris-Lavine theorem is discussed.

Contents

Chapter I

Introduction

SUMMARY OF THE PRESENT CHAPTER

I.1 BACKGROUND

This thesis revolves around the random Schrödinger operator, i.e., differential operator taking the form

$$\mathcal{H} = (i\nabla + \mathbf{A})^2 + V, \tag{I.1}$$

where $\mathbf{A}$ is a vector-valued function and V is a random scalar potential. The interest on such operators goes back to the axioms of Quantum Mechanics, where physical quantities of a given system, such as the energy, are postulated to be represented by self-adjoint differential operators, and what one can really measure out of these quantities in experiments correspond to the spectral values of the associated operators. The procedure of associating a differential operator to a physical quantity is called *quantization*. Indeed, the operator of form (I.1) is obtained by quantizing the Hamiltonian (i.e., total energy) of a single particle moving in a magnetic field $\mathbf{B} = \operatorname{curl} \mathbf{A}$ and electric potential V.

Needless to say, it is of great interest for physicists to understand the spectral properties of (I.1).

To properly define (I.1) as a self-adjoint operator is not a trivial task. In mathematical terms, it consists in choosing a Hilbert space $\mathscr{H}$ (usually L^2) and a suitable subspace $\mathscr{D} \subset \mathscr{H}$ in such a way that the linear map $\mathcal{H} : \mathscr{D} \to \mathscr{H}$ is self-adjoint. This type of operator is called *unbounded* due to the fact that its differential expression (I.1) is only expected to make sense on a proper subspace $\mathscr{D}$ of $\mathscr{H}$, as opposed to the case of bounded operators. The nature of operator $\mathcal{H}$ varies greatly with the given $\mathscr{D}$, in the sense that even if the expression (I.1) could be defined on two distinct subspaces $\mathscr{D}_1$ and $\mathscr{D}_2$, the spectral properties of $\mathscr{H}$ on each domain could be completely different. This matter of domain is more than a mathematical consideration: physics usually imposes constraints on the choice of $\mathscr{D}$.

Given a self-adjoint operator $\mathcal{H}$, the spectral theorem (Theorem I.12) and functional calculus (Theorem I.11) allows to solve the Schrödinger equation

$$i\partial_t u = \mathcal{H}u, \quad u(0,\cdot) = \psi \in \mathscr{H},$$

whose solution $u = e^{-it\mathcal{H}}\psi$ describes the time-evolution of the particle's wave function (with initial condition ψ) in the system associated to the Hamiltonian $\mathcal{H}$. The behaviour of u is in fact determined by the spectral nature of $\mathcal{H}$. More precisely, the spectrum of a self-adjoint operator $\mathcal{H}$ is a closed set $\sigma(\mathcal{H}) \subset \mathbb{R}$ which can be decomposed into

$$\sigma(\mathcal{H}) = \overline{\sigma_{\mathrm{pp}}(\mathcal{H})} \cup \sigma_{\mathrm{c}}(H).$$

Here, $\sigma_{\mathrm{pp}}(\mathcal{H})$ denotes the pure point spectrum, i.e., the collection of eigenvalues of $\mathcal{H}$, while $\sigma_{\mathrm{c}}(\mathcal{H})$ represents the continuous spectrum. Each of these spectra corresponds to a subspace of $\mathscr{H}$ satisfying

$$\mathscr{H} = \mathscr{H}_{\mathrm{pp}} \oplus \mathscr{H}_{\mathrm{c}}.$$

Their precise definitions will not be given, however one can keep in mind the following picture given by the famous RAGE theorem: the solution u to the Schrödinger equations with a initial wave function $\psi \in \mathscr{H}_{\mathrm{pp}}$ will stay concentrated within a bounded region during the time evolution. On the other hand, the mass of u corresponding to a initial wave function $\psi \in \mathscr{H}_{\mathrm{c}}$ will spread out to the whole space. In physicists' terminology, the spectral values in σ_{pp} are referred to as the *bound states* while those in σ_{c} are referred to as the *scattering states* or *extended states*[1]. As a matter of fact, the understanding of the spectral nature of a given Schrödinger operator is of primordial importance to describe the dynamics of a quantum system.

The need for randomness in V goes back to the study of disordered systems, one of the most prominent examples of which is the model of Anderson [And58] who aims

[1] In some literature, the scattering states only refer to the *absolutely continuous spectrum* which is only a subset of the σ_c; the physical meaning of the remaining part, the singularly continuous spectrum, is subtle and beyong the scope of the thesis.

to explain the loss of conductivity when impurities are present in a metallic conductor. Anderson considered the Hamiltonian

$$-\Delta_{\mathrm{d}} + \eta \tag{I.2}$$

on the Hilbert space $\ell^2(\mathbb{Z}^d)$ where Δ_{d} denotes the discrete Laplacian on the lattice $\mathbb{Z}^d$ and $\eta = (\eta(x))_{x\in\mathbb{Z}^d}$ is a collection of i.i.d. random variables. It is believed that Anderson's model has the following spectral properties, now known as *Anderson localization*:

- In dimensions $d = 1, 2$, (I.2) has pure point spectrum $\sigma = \overline{\sigma_{\mathrm{pp}}}$ with exponentially decaying eigenfunctions.
- In dimension $d \geq 3$, the localization holds for sufficiently low energies, i.e., $\sigma \cap (-\infty, E]$ should be pure point with exponentially decaying eigenfunctions for sufficiently low E.

The above prediction gives the following picture: In low dimensions $d = 1, 2$, once the randomness is turned on (no matter of what intensity), the wave function will stay localized and thus the conductivity vanishes; for dimensions $d \geq 3$, the conductivity vanishes provided that the energy is low enough with respect to the disorder strength. Anderson localization has sparked great interest in mathematical community, and in the discrete case described above, mathematically rigorous proofs have been obtained under various settings. For a detailed survey in the discrete case, see the lecture notes by Kirsch [Kir07] and the references therein.

Efforts have also been made to generalize the discrete settings to continuous ones. On the Hilbert space $\mathscr{H} = L^2(\mathbb{R}^d)$, we consider the operator (I.1) with random potential V defined on $\mathbb{R}^d$. A natural choice for the potential in this case is the stationary Gaussian field characterized by the covariance function $C(x) = \mathbb{E}[V(0)V(x)]$, $x \in \mathbb{R}^d$. Due to the unboundedness of V, one first needs to argue that such operator is self-adjoint on some domain: this is proved by [KM83].

The objective of this thesis is to extend the scope of continuous setting to include distribution-valued random potentials, and, as a long-term goal, to see if Anderson's prediction described above persists in this case. A major example of this class is the continuous Anderson Hamiltonian

$$\mathcal{H} = -\Delta + \xi \tag{I.3}$$

which arises as the formal scaling limit of the discrete operator (I.2). Here, Δ is the Laplacian on $\mathbb{R}^d$ and ξ the spatial Gaussian white noise. More precisely, the Gaussian white noise ξ is the limit in law of the rescaled random field $\xi_h(x) = h^{d/2}\eta(x/h)$, $x \in h\mathbb{Z}^d$, as $h \to 0$, where $(\eta(k))_{k\in\mathbb{Z}^d}$ is the i.i.d. family of random variables given in (I.2). The white noise can also be seen as a stationary Gaussian field with δ-covariance, $C(x) = \delta(x)$. Of course, this statement should be interpreted in the sense of distribution: ξ is the Gaussian family $(\xi(\varphi))$ indexed by smooth compactly supported test functions $\varphi \in C_c^\infty(\mathbb{R}^d)$ with the covariance function

$$\mathbb{E}[\xi(\varphi_1)\xi_1(\varphi_2)] = \langle \varphi_1, \varphi_2 \rangle_{L^2}\,.$$

Indeed, the right-hand side of the above identity is the rigorous definition of $\iint \varphi(x)\delta(x-y)\varphi(y)\,\mathrm{d}x\,\mathrm{d}y$ which justifies the formal expression $C = \delta$. In fact, the Gaussian white noise ξ is only distribution-valued and almost surely has a negative Hölder regularity $-d/2-\varepsilon$, with d being the spatial dimension (see Definition I.24 and Remark I.25). Due to this singularity, one can only define the product $\xi \cdot f$ if the function f is smooth enough (actually, smoother than $(d/2+\varepsilon)$-Hölder). For this reason, classical operator theory fails to identify a domain on which (I.3) is well-defined, and therefore a mathematically rigorous meaning of (I.3) as a self-adjoint operator has been inaccessible until only recently. We shall call these classically ill-posed operators by the term *singular random Schrödinger operators* (Definition I.26).

A way to give mathematically rigorous sense to these operators is to utilize the novel theories of singular SPDE. Two of the most important breakthroughs in the field are the theory of regularity structures [Hai14] due to Hairer and the theory of paracontrolled distributions due to Gubinelli, Imkeller and Perkowski [GIP15]. In section I.3.2 we will explain the SPDE methodology to construct singular operators and readers can find in section I.3.4 a survey of recent advancement in the business. Particularly, the theory of regularity structures plays an essential role in the present thesis. For this reason, we devote the section I.4 to its introduction: the aim is to illustrate its framework (Figure I.1) and application to construct singular random Schrödinger operators.

Another consideration in our setting is the presence of the magnetic vector potential $\mathbf{A}$. The motivation to include the magnetic term comes from the drastic difference in spectrum between the operator $(i\nabla + \mathbf{A})^2$ and the Laplacian $-\Delta$ on full space $\mathbb{R}^d$. Here, the former operator is called the *Landau Hamiltonian* or the *magnetic Laplacian* as it is resulted from the quantization of the kinetic energy of a particle traveling in a magnetic field. For instance, if we consider a particle on a xy-plane on which a magnetic field of magnitude $B > 0$ pointing at the z-direction is applied, then the magnetic potential $\mathbf{A}$ can be set to[2]

$$\mathbf{A}(x_1, x_2) = \frac{B}{2}(-x_2, x_1), \quad (x_1, x_2) \in \mathbb{R}^2. \tag{I.4}$$

With this $\mathbf{A}$, the spectrum of Landau Hamiltonian $(i\nabla + \mathbf{A})^2$ as an unbounded operator on $L^2(\mathbb{R}^2)$ is given by

$$\{(2n+1)B : n = 0, 1, 2, \dots\},$$

where each $(2n+1)B$ is an eigenvalue of infinite multiplicity, called a *Landau level*. As one can see, this spectrum is very different from that of the Laplacian $-\Delta$, which coincides with the non-negative real line $[0, +\infty)$. Therefore, a natural question arises: how would the perturbation by a white noise impact the spectrum of Landau Hamiltonian? Our objective is to start the study in this direction.

[2] Note that, given a magnetic field $\mathbf{B}$, the choice of a corresponding $\mathbf{A}$ is not unique. Indeed, if $\operatorname{curl}\mathbf{A} = \mathbf{B}$, then for any two-times differentiable scalar field f, $\operatorname{curl}(\mathbf{A} + \nabla f) = \mathbf{B}$. However, these different choices of $\mathbf{A}$ all lead to unitarily equivalent operators, and therefore the same *physics*.

In this thesis, we will only consider magnetic vector potential of the form (I.4) for dimension $d = 2$, or its analogue for $d = 3$,

$$\mathbf{A}(x_1, x_2, x_3) = \frac{B}{2}(-x_2, x_1, 0), \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \tag{I.5}$$

although we could probably allow much more general potentials such as those in $L^2_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^d)$. For readers feeling uneasy with the magnetic field, one can just put $\mathbf{A} = 0$ in the majority part of this thesis, in which case $(i\nabla + \mathbf{A})^2$ is just reduced to the Laplacian $-\Delta$.

Let us point out that it is more interesting to consider the magnetic operator on full Euclidean space than on a bounded volume space such as the torus $\mathbb{T}^d$ (which would be much more amenable to analysis by SPDE theories): for unperturbed Landau Hamiltonian with $\mathbf{A}$ given by (I.4), (I.5), the Landau levels only arise on the full space; on a bounded volume space, $-\Delta$ and $(i\nabla + \mathbf{A})^2$ have equivalent quadratic forms and both have purely discrete spectrum with eigenvalues of finite multiplicity.

The remaining of this chapter is organized as follows:
We begin by a crash presentation on the spectral theory of unbounded operators in section I.2, where we introduce necessary notions for the thesis. In section I.3, the concept of random operators, or operator-valued random variables, is introduced; in particular, we elaborate how SPDEs enter the game by starting from the basic case of bounded continuous random potential and discussing technicalities in more advanced settings. In section I.4, we present the theory of regularity structures with an emphasis on the motivations and ideas; the aim is to illustrate how the theory allows to construct a random operator with white noise potential. Finally, in section I.5 we present an overview to the contributions of this thesis and elaborate the central ideas of each chapter in the hope to facilitate the subsequent reading.

I.2 THEORY OF UNBOUNDED OPERATORS

The section aims at giving a brief but self-contained account on notions of spectral theory which will be used in the sequel. Useful references are [Lew22, Sch12, Tes14, RS80].

I.2.1 From basic notions to spectral theorem

An (unbounded) operator is a linear map acting on a certain Hilbert space. As a prototypical example, one can think of the Laplacian operator Δ acting on the Hilbert space $L^2(\mathbb{R})$. One sees that Laplacian is indeed linear; however, it does not act on every element of $L^2(\mathbb{R})$, but only a dense subspace such as compactly supported smooth functions $C_c^\infty(\mathbb{R})$ or Sobolev space $H^2(\mathbb{R})$. For this reason, we do not expect to define the operator everywhere over $\mathscr{H}$ (hence the adjective *unbounded*). We formalize this in the following definition:

Definition I.1. Given a seperable Hilbert space $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ over $\mathbb{C}$ and a linear subspace $\mathcal{D}(A)$, we say that the map $A : \mathcal{D}(A) \subset \mathscr{H} \to \mathscr{H}$ is an (unbounded) operator, if A is linear on $\mathcal{D}(A)$. In this case, we say that $\mathcal{D}(A)$ is the domain of A and we will use the notation $(A, \mathcal{D}(A))$ to specify the action and the domain of the operator.

A way to characterize an operator is to look at its graph, that is, the subspace of $\mathscr{H} \times \mathscr{H}$ given by $\mathcal{G}(A) = \{(f, Af) : f \in \mathcal{D}(A)\}$. An operator is defined if and only if a graph is given, and a linear subspace $G \subset \mathscr{H} \times \mathscr{H}$ coincides with the graph of an operator if and only if

$$(0, g) \in G \implies g = 0. \tag{I.6}$$

The concept of graph allows us to study the basic properties of the operator.

Definition I.2. Given an operator $(A, \mathcal{D}(A))$,

- A is said to be **closed** if $\mathcal{G}(A)$ is closed in $\mathscr{H} \times \mathscr{H}$. That is, for every sequence $(f_n) \subset \mathcal{D}(A)$ and $g \in \mathscr{H}$ such that $f_n \to f$ and $Af_n \to g$ in $\mathscr{H}$, we have $f \in \mathcal{D}(A)$ and $Af = g$.
- A is said to be **closable** if the closure of $\mathcal{G}(A)$ satisfies the property (I.6). In this case, $\overline{\mathcal{G}(A)}$ coincides with the graph of a operator $\bar{A}$, called the closure of A.

In a sense, closedness is the replacement for continuity in the regime of unbounded operators, as one can see that if an operator A is defined everywhere (i.e. $\mathcal{D}(A) = \mathscr{H}$) and is closed, then A is bounded (closed graph theorem).

For a given operator $(A, \mathcal{D}(A))$, we want to find a counterpart operator $(A^*, \mathcal{D}(A^*))$ (for a largest possible $\mathcal{D}(A^*)$), named the **adjoint** of A, satisfying the following relation

$$\langle Af, g \rangle = \langle f, A^* g \rangle, \quad \forall f \in \mathcal{D}(A),\ g \in \mathcal{D}(A^*),$$

or equivalently $\langle (g, A^*g), (Af, -f) \rangle_{\mathscr{H} \times \mathscr{H}} = 0$. Therefore, if there is such an operator A^* at all, its graph should verify

$$\mathcal{G}(A^*) = \{(Af, -f) : f \in \mathcal{D}(A)\}^{\perp}, \tag{I.7}$$

where $\perp$ denotes the orthogonal complement in $\mathscr{H} \times \mathscr{H}$. However, the right-hand side of (I.7) does not necessarily define a graph (for which one needs the condition (I.6)). Indeed, it holds

$$[(0, g) \in \mathcal{G}(A^*) \implies g = 0] \iff \mathcal{D}(A) \text{ is dense.}$$

This means that in order to properly define the adjoint A^*, one needs A to be densely defined, i.e. the domain $\mathcal{D}(A)$ is a dense subspace of $\mathscr{H}$. We summarize the above discussion in the following definition

Definition I.3. For a *densely defined* operator $(A, \mathcal{D}(A))$, the adjoint A^* is the operator corresponding to the graph (I.7). Equivalently, by (I.7), the domain of A^* is given by

$$\mathcal{D}(A^*) = \{f \in \mathscr{H} : \exists g_f \in \mathscr{H}, \forall h \in \mathcal{D}(A),\ \langle g_f, h \rangle = \langle f, Ah \rangle\}$$

and $A^* f = g_f$ where g_f is the element in $\mathscr{H}$ associated to f in the above identity.

As a corollary, an adjoint A^* is always closed, since its graph is an orthogonal complement which is always closed in $\mathscr{H} \times \mathscr{H}$. On the other hand, it is not always true that A^* is densely defined. However, we have the following

Proposition I.4. *$\mathcal{D}(A^*)$ is dense if and only if A is closable. In this case, we have $(A^*)^* = \bar{A}$.*

Proof. It holds that

$$w \in \mathcal{D}(A^*)^\perp \iff (w, 0) \in \mathcal{G}(A^*)^\perp = \overline{\{(Af, -f) : f \in \mathcal{D}(A)\}} \iff (0, w) \in \overline{\mathcal{G}(A)}.$$

So $\mathcal{D}(A^*)$ is dense if and only if $\overline{\mathcal{G}(A)}$ is a graph, i.e. A closable. □

Now we study the concept of spectrum and resolvent.

Definition I.5. A complex number $z \in \mathbb{C}$ is a **resolvent value** of the operator A if $A - z$ is bijective from $\mathcal{D}(A)$ into $\mathscr{H}$ and admits a bounded inverse. The collection of resolvent values is called the resolvent set of A and is denoted by $\rho(A)$. The **spectrum** of A is defined to be the complement of $\rho(A)$, denoted by $\sigma(A)$. Elements of $\sigma(A)$ are called **spectral values**, and a spectral value $z \in \sigma(A)$ is an **eigenvalue** if $A - z$ is not injective, i.e. $\mathrm{Ker}(A - z) \neq \{0\}$.

The following proposition shows that if an operator is not closed, it will have a trivial spectrum.

Proposition I.6. *If an operator A is not closed, then its spectrum is the whole space $\sigma(A) = \mathbb{C}$.*

Proof. By contraposition we show that A is closed if $\rho(A)$ is non-empty. Pick $z \in \rho(A)$ and let $f, g \in \mathscr{H}$ and $(f_n) \subset \mathcal{D}(A)$ be such that $f_n \to f$ and $Af_n \to g$. Set $u_n = (A - z)f_n$. Then, we have $u_n \to g - zf$ in $\mathscr{H}$. Since z is a resolvent value, $(A - z)^{-1} : \mathscr{H} \to \mathscr{D}(A)$ is bounded and we can deduce

$$f = \lim_n f_n = \lim_n (A - z)^{-1} u_n = (A - z)^{-1}(g - zf),$$

which implies $f \in \mathcal{D}(A)$ and that $Af = g$. □

The class of operators we will concentrate on is the class of self-adjoint operators.

Definition I.7.

- We say an operator A is symmetric if

$$\langle f, Ag \rangle = \langle Af, g \rangle, \quad \forall f, g \in \mathcal{D}(A).$$

  This is equivalent to say A^* is an extension of A, i.e. $\mathcal{G}(A) \subset \mathcal{G}(A^*)$.

- We say an operator A is self-adjoint if $A = A^*$, i.e. $\mathcal{G}(A) = \mathcal{G}(A^*)$.

By the above definition, every self-adjoint operator is closed. This implies every symmetric operator A is closable. Indeed, since A^* is closed, one has $\overline{\mathcal{G}(A)} \subset \mathcal{G}(A^*)$ and thus (I.6) is automatically satisfied for $\overline{\mathcal{G}(A)}$.

Below is an important criterion for a symmetric operator A to be self-adjoint.

Proposition I.8. *Let A be symmetric. Then the following assertions are equivalent:*

1. *A is self-adjoint.*
2. *$\sigma(A) \subset \mathbb{R}$.*
3. *There exists some $\lambda \in \mathbb{C}$ such that $A - \lambda$, $A - \lambda^*$ are surjective. (Here λ can be real.)*

Proof. Since A is symmetric, $\langle Au, u \rangle$ is real for any $u \in \mathcal{D}(A)$. Thus for $z = a + ib$, we have

$$\|(A-z)u\|^2 = \|(A-a)u\|^2 + |b|^2 \|u\|^2 - 2\,\mathrm{Re}\,\langle (A-a)u, ibu \rangle = \|(A-a)u\|^2 + |b|^2 \|u\|^2$$

giving the bound

$$|b|^2 \|u\|^2 \leq \|(A-z)u\|^2 . \tag{I.8}$$

(1. $\Longrightarrow$ 2.) Now suppose that A is self-adjoint and fix $b \neq 0$. Then (I.8) implies that $A - z$ and $A - z^*$ are injective. By self-adjointness and the fact $\mathrm{Ran}(A-z)^\perp = \mathrm{Ker}(A^* - z^*)$, this also entails

$$\mathrm{Ran}(A-z)^\perp = \mathrm{Ker}(A - z^*) = \{0\}.$$

That is, $\mathrm{Ran}(A-z)$ is dense. Now observe that $\mathrm{Ran}(A-z)$ is also closed. Indeed, let $g \in \mathscr{H}$ be a limit point of $\mathrm{Ran}(A-z)$, then there is a sequence $(f_n) \subset \mathcal{D}(A)$ such that $(A-z)f_n$ converges to g. It follows from (I.8) that (f_n) is Cauchy. Let us denote by f the limit of (f_n) in $\mathscr{H}$. Since A is self-adjoint (and therefore closed), we must have $f \in \mathcal{D}(A)$ and $g = (A-z)f \in \mathrm{Ran}(A-z)$. Together, these imply that $\mathrm{Ran}(A-z) = \overline{\mathrm{Ran}(A-z)} = \mathscr{H}$, i.e. $A - z$ is surjective.
The above proves that $A - z$ is bijective. Furthermore, (I.8) shows the inverse $(A-z)^{-1}$ is bounded, from which one deduces $a + ib \in \rho(A)$ for all $b \neq 0$.

(2. $\Longrightarrow$ 3.) Obvious. Choose for example $\lambda = i$.

(3. $\Longrightarrow$ 1.) Since A is symmetric, we only need to show $\mathcal{D}(A) \supset \mathcal{D}(A^*)$. Fix $v \in \mathcal{D}(A^*)$. Recall from Definition I.3 that there exists some $w \in \mathscr{H}$ such that

$$\langle v, Ah \rangle = \langle w, h \rangle , \quad \forall h \in \mathcal{D}(A).$$

Let λ be the complex number given in the assumption. We deduce that $\langle v, (A-\lambda)h \rangle = \langle w - \lambda^* v, h \rangle$ for every $h \in \mathcal{D}(A)$. As $w - \lambda^* v \in \mathscr{H}$, the surjectivity of $A - \lambda^*$ and the symmetry of A together imply the existence of some $y \in \mathcal{D}(A)$ such that

$$\langle v, (A-\lambda)h \rangle = \langle (A-\lambda^*)y, h \rangle = \langle y, (A-\lambda)h \rangle$$

i.e. $\langle v - y, (A-\lambda)h \rangle = 0$ for any $h \in \mathcal{D}(A)$. With the assumption that $A - \lambda$ is surjective, this shows nothing but $v - y \in \mathrm{Ran}(A-\lambda)^\perp = \{0\}$, that is $v = y \in \mathcal{D}(A)$. Therefore, one has $\mathcal{D}(A^*) \subset \mathcal{D}(A)$ and concludes the proof. ☐

The following is a crucial criterion for a given $\lambda \in \mathbb{R}$ to be a spectral value of a self-adjoint operator, which will be used in the following chapters.

Theorem I.9 (Weyl's criterion)**.** *Given a self-adjoint operator A and a real number $\lambda \in \mathbb{R}$, $\lambda \in \sigma(A)$ if and only if λ admits a **Weyl's sequence**: a sequence $(f_n) \subset \mathcal{D}(A)$ such that $\|f_n\| = 1$ for all n and $Af_n - \lambda f_n \to 0$ in $\mathscr{H}$ as $n \to \infty$.*

Proof. Suppose first that λ admits a Weyl's sequence (f_n). Then λ cannot be a resolvent value, simply because even if $A-\lambda$ is bijective, its inverse will not be bounded: Assuming $(A-\lambda)^{-1}$ exists and is bounded by some constant C, we have

$$1 = \|f_n\| = \left\|(A-\lambda)^{-1}(A-\lambda)f_n\right\| \leq C\,\|(A-\lambda)f_n\|\,,$$

which leads to a contradiction as the right-hand side converges to 0.

For the other implication, we argue by contraposition: Suppose λ does not admit a Weyl's sequence, we show that λ is a resolvent value. Since a Weyl's sequence does not exist, one must have

$$\|(A-\lambda)f\| \geq \varepsilon\,\|f\|\,, \quad \forall f \in \mathcal{D}(A) \tag{I.9}$$

for some $\varepsilon > 0$. (Otherwise one would have $\inf\{\|(A-\lambda)f\| : f \in \mathcal{D}(A), \|f\| = 1\} = 0$ which implies the existence of a Weyl's sequence.) As in the proof of Theorem I.8 (1. $\Longrightarrow$ 2.), this implies that $A-\lambda$ is bijective with bounded inverse. Consequently, $\lambda \in \rho(A)$. $\square$

Before stating the spectral theorem for self-adjoint operators, we need to introduce the concept of **projection-valued measure**.

Definition I.10. A **projection-valued measure** (or **spectral resolution** in some literature) is a set function $E : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathscr{H})$ such that

1. For every Borel set $B \in \mathcal{B}(\mathbb{R})$, $E(B)$ is an orthogonal projection.
2. $E(\varnothing) = 0$ and $E(\mathbb{R}) = I$.
3. E is strongly σ-additive, i.e. for a sequence of pairwise disjoint Borel sets (B_n),

$$E\left(\bigcup_n B_n\right) = \text{s-}\lim_{N\to\infty} \sum_{n=0}^{N} E(B_n),$$

where s- lim denotes the strong limit of bounded operators.

Note that for every $x, y \in \mathscr{H}$, $\mu_{x,y}(\cdot) = \langle E(\cdot)x, y\rangle$ defines a complex-valued measure whose total variation satisfies

$$|\mu_{x,y}|(B) \leq \mu_x(B)^{1/2}\mu_y(B)^{1/2},$$

where $\mu_x(\cdot) = \langle E(\cdot)x, x\rangle$, $\mu_y(\cdot) = \langle E(\cdot)y, y\rangle$ are Borel-measures with total mass $\|x\|^2$, $\|y\|^2$, respectively.

Given a projection valued measure, we can imitate Lebesgue's integration theory to define a **functional calculus**. That is, to construct an integral which to every Borel-measurable function associates an unbounded operator. Let us outline the reasoning as follows:

I. For any simple functions $f = \sum_{j=1}^{n} c_j \mathbf{1}_{B_j}$, where B_j's are disjoint Borel sets and c_j's coefficients in $\mathbb{C}$, we pose

$$\mathbf{I}(f) = \int_{\mathbb{R}} f(\lambda)E(\mathrm{d}\lambda) = \sum_{j=1}^{n} c_j E(B_j)$$

Note that $\mathbf{I}(f)$ is bounded and $\|\mathbf{I}(f)\| \leq \|f\|_\infty$.

II. For any bounded $\mathcal{B}(\mathbb{R})$-measurable function f, it can be approximated by a sequence of simple functions (f_n). As $(\mathbf{I}(f_n))_n$ forms a Cauchy sequence in the space of bounded operators on $\mathscr{H}$, one thus defines

$$\mathbf{I}(f) = \lim_{n\to\infty} \mathbf{I}(f_n).$$

One can show that $\mathbf{I}(f)$ does not depend on the choice of approximating sequence (f_n).

III. For any $\mathcal{B}(\mathbb{R})$-measurable function which is finite E-a.e., that is $E(\{\lambda \in \mathbb{R} : f(\lambda) = \infty\}) = 0$, one can introduce a bounding sequence of sets $(M_n) \subset \mathcal{B}(\mathbb{R})$ such that $M_n \subset M_{n+1}$, $E(\cup_n M_n) = I$ and f is bounded on each M_n. We hence define the unbounded operator $\mathbf{I}(f) = \int_{\mathbb{R}} f(\lambda)E(\mathrm{d}\lambda)$ as

$$\mathcal{D}(\mathbf{I}(f)) = \left\{x \in \mathscr{H} : \int_{\mathbb{R}} |f(\lambda)|^2 \langle E(\mathrm{d}\lambda)x, x\rangle < \infty.\right\},$$
$$\mathbf{I}(f)x = \lim_{n\to\infty} \mathbf{I}(f\mathbf{1}_{M_n})x, \quad x \in \mathcal{D}(\mathbf{I}(f)).$$

One can show $\mathbf{I}(f)$ is independent of the choice of (M_n), and that $\mathbf{I}(f)$ is densely defined since for all $x \in \mathscr{H}$, one has $x = \lim_{n\to\infty} E(M_n)x$ and $E(M_n)x$ is in $\mathcal{D}(\mathbf{I}(f))$ for all n.

The functional calculus establishes a connection between complex-valued measurable functions and unbounded operators.

Theorem I.11 ([Sch12]). *For all E-a.e. finite Borel-measurable functions f, g and $\alpha, \beta \in \mathbb{C}$, one has*

1. $\mathbf{I}(f^*) = \mathbf{I}(f)^*$, *where f^* denotes the complex-conjugate of f. In particular, $\mathbf{I}(f)$ is a closed operator for all f.*
2. $\mathbf{I}(\alpha f + \beta g) = \overline{\alpha\mathbf{I}(f) + \beta\mathbf{I}(g)}$.
3. $\mathbf{I}(fg) = \overline{\mathbf{I}(f)\mathbf{I}(g)}$.

4. $\mathcal{D}(\mathbf{I}(f)\mathbf{I}(g)) = \mathcal{D}(\mathbf{I}(g)) \cap \mathcal{D}(\mathbf{I}(fg))$.

In particular, if f is real-valued, then $\mathbf{I}(f)$ is automatically self-adjoint. This shows that functional calculus is a powerful tool: given a projection-valued measure, one can generate self-adjoint operator $\mathbf{I}(f)$ for any real-valued, measurable, E-a.e. finite f.

Now we are ready to state the spectral theorem, which is the converse of the previous statement. In the sequel, for any given projection-valued E and any E-a.e. finite function, we should denote $\int f(\lambda)E(\mathrm{d}\lambda) = \mathbf{I}(f)$.

Theorem I.12 (Spectral theorem). *Let A be a self-adjoint operator on the Hilbert space $\mathscr{H}$. Then there exists a unique projection-valued measure E_A such that*

$$A = \int_{\mathbb{R}} \lambda E_A(\mathrm{d}\lambda).$$

Sketch of proof. The idea is to first consider the special case of bounded operators, and then generalize to unbounded case.

(A bounded) For A is bounded, one can define the bounded operator $p(A)$ for all complex polynomial p. Now fix two elements $x, y \in \mathscr{H}$, one observes that $p \mapsto \langle p(A)x, y\rangle$ defines a continuous functional on the space of polynomials equipped with the norm $\|p\|_J = \sup\{|p(t)| : t \in J\}$, where J is a compact interval containing $\sigma(A)$. Stone-Weierstrass theorem then allows us to extend continuously the functional to all $p \in C(J)$. In turn, Riesz representation theorem implies the existence of a measure $\mu_{x,y}$ such that for all $p \in C(J)$,

$$\langle p(A)x, y\rangle = \int_J p(\lambda)\mu_{x,y}(\mathrm{d}\lambda)$$

By the uniqueness of representation theorem, it is not hard to see that $\mu_{x,y}$ is linear in both x and y; furthermore, for any fixed $B \in \mathcal{B}(J)$, $(x, y) \mapsto \mu_{x,y}(B)$ defines a continuous sesquilinear form on $\mathscr{H}$. Linear space version of Riesz representation then shows the existence of bounded operator $E_A(B)$ such that $\langle E_A(B)x, y\rangle = \mu_{x,y}(B)$. The fact that $E_A(B)$ is an orthogonal projection follows from the multiplicative property $pq(A) = p(A)q(A)$ for all polynomials p, q. Up to now we have shown E_A is indeed a projection-valued measure. Finally, let $p_0(t) = t$, one has $\langle \mathbf{I}(p_0)x, y\rangle = \langle Ax, y\rangle$ for all $x, y \in \mathscr{H}$, which proves $A = \mathbf{I}(p_0) = \int_J \lambda E_A(\mathrm{d}\lambda)$. The uniqueness part of the theorem essentially follows from the uniqueness of Riesz representation: If E, F are two projection-valued measures satisfying the consequence of the theorem, then they define the same continuous functional on polynomials.

(A unbounded) For an unbounded self-adjoint A, the key observation is that $Z = A(1 + A^2)^{-1/2}$ defines a bounded self-adjoint operator whose spectrum is contained in the interval $[-1, 1]$. We apply the previous case to derive a projection-valued measure E_Z associated to Z. Heuristically, we want to recover A using the formal identity $A = Z(1 - Z^2)^{-1/2}$. This idea can be made sense of using functional calculus: indeed, one can show that the function $\varphi(z) = z(1-z^2)^{-1/2}$ is finite E_Z-a.e. and that A coincides with $\int \varphi\, \mathrm{d}E_Z$. To conclude, we pose $E_A(B) = E_Z(\varphi^{-1}(B))$; in particular, E_A is indeed a projection-valued measure and one has

$$\int_{\mathbb{R}} \lambda E_A(\mathrm{d}\lambda) = \int_{[-1,1]} \varphi(z)E_Z(\mathrm{d}z) = A$$

The uniqueness follows from the uniqueness in the bounded case. □

I.2.2 Quadratic form and Friedrichs' extension

In practice, to find a self-adjoint operator representing a quantum system, we usually start with an operator acting on a sufficiently small domain and then try to find a self-adjoint extension. For example, when dealing with a Schrödinger operator $A = -\Delta + V$, a common procedure is to consider A over the domain C_c^∞ on which the symmetry usually follows from integration by parts; one then proceeds to find a self-adjoint extension of (A, C_c^∞). As a result, we are in need of an argument which guarantees the existence of such an extension.

However, there are situations where A is not even well-defined on C_c^∞, for instance when V has a singularity of $|x|^{-\alpha}$ with $\alpha \geq d/2$ at the origin, in which case $(-\Delta + V)\varphi$ is not a L^2-function whenever $\varphi \in C_c^\infty$ does not vanish at 0. On the other hand, provided that $\alpha < d$, one notices that the associated quadratic form of A, $q_A(\varphi) = \langle (-\Delta + V)\varphi, \varphi \rangle$ is finite. It implies that, in the regime $d/2 \leq \alpha < d$, the operator A itself does not make sense over C_c^∞, but the quadratic form q_A is however well-defined.

Now the question is that, given a quadratic form, is it always possible to find an associated self-adjoint operator? Let us denote also by q_A the bilinear form $q_A(\varphi, \psi) = \langle (-\Delta + V)\varphi, \psi \rangle$. Suppose that q_A is well-defined on C_c^∞ (i.e. $q_A(\varphi, \varphi) < \infty$ for all $\varphi \in C_c^\infty$). Since V is real-valued, integration by parts shows that q_A is symmetric. Assume now that q_A satisfies the bound, $q_A(\varphi, \varphi) \geq c \|\varphi\|^2$ for some $c > 0$. Then it follows that the set C_c^∞ can be completed by the norm $q_A(\cdot)^{1/2}$, resulting in a Hilbert subspace $\mathscr{Q} \subset \mathscr{H}$, equipped with the scalar product $\langle \cdot, \cdot \rangle_{\mathscr{Q}} = q_A(\cdot, \cdot)$. The following theorem due to Friedrichs provides an answer to the question.

Theorem I.13 (Friedrichs). *Let $(\mathscr{Q}, \langle \cdot, \cdot \rangle_{\mathscr{Q}})$ and $(\mathscr{H}, \langle \cdot, \cdot \rangle)$ be two Hilbert spaces. Assume $\mathscr{Q}$ is a dense subspace of $\mathscr{H}$ and that $\mathscr{Q}$ embeds continuously in $\mathscr{H}$, that is, there exists $c > 0$ such that $c\|v\|^2 \leq \|v\|_{\mathscr{Q}}^2$ for all $v \in \mathscr{Q}$.*

Then, the set

$$\mathcal{G} := \{(u, w) \in \mathscr{Q} \times \mathscr{H} : \forall h \in \mathscr{Q}, \langle u, h \rangle_{\mathscr{Q}} = \langle w, h \rangle_{\mathscr{H}}\} \tag{I.10}$$

defines the graph of an operator A on $\mathscr{H}$ with domain $\mathcal{D}(A) = \{u \in \mathscr{Q} : \exists w \in \mathscr{H}, (u, w) \in \mathcal{G}\}$. Moreover, the operator A satisfies the following properties

- *$\mathcal{D}(A)$ is dense in $(\mathscr{Q}, \langle \cdot, \cdot \rangle_{\mathscr{Q}})$ and therefore dense in $(\mathscr{H}, \langle \cdot, \cdot \rangle)$.*
- *$(A, \mathcal{D}(A))$ is self-adjoint.*
- *$q_A(u) = \|u\|_{\mathscr{Q}}^2$ for all $u \in \mathcal{D}(A)$, where $q_A(u) := \langle Au, u \rangle$.*
- *If B is another self-adjoint operator whose domain $\mathcal{D}(B)$ is dense in $\mathscr{Q}$ such that $q_B(u) = \|u\|_{\mathscr{Q}}^2$ for all $u \in \mathcal{D}(B)$, then $A = B$.*

Proof. Let us verify (I.10) defines indeed the graph of an operator. Suppose $(0, w) \in \mathcal{G}(A)$. By definition, we have $\langle w, h \rangle_{\mathscr{H}} = \langle 0, h \rangle_{\mathscr{Q}} = 0$ for all $h \in \mathscr{Q}$. Since $\mathscr{Q}$ is dense in $\mathscr{H}$, one deduces $w = 0$. Using (I.6), it follows that $\mathcal{G}$ is the graph of an operator A.

By construction, it is easy to see that A is symmetric: for all $u, v \in \mathcal{D}(A)$,

$$\langle u, Av \rangle = \langle Av, u \rangle^* = \langle v, u \rangle_{\mathscr{Q}}^* = \langle u, v \rangle_{\mathscr{Q}} = \langle Au, v \rangle .$$

Note also that A is surjective: for all $w \in \mathscr{H}$, the map $\mathscr{Q} \ni h \mapsto \langle w, h \rangle_{\mathscr{H}}$ is linear and continuous on $\mathscr{Q}$. By the Riesz representation, there exists some (unique) $u \in \mathscr{Q}$ such that $\langle u, h \rangle_{\mathscr{Q}} = \langle w, h \rangle_{\mathscr{H}}$. By the definition of A, this means nothing but $(u, w) \in \mathcal{G}$, or $u \in \mathcal{D}(A)$, and indeed A is surjective.

We check now $\mathcal{D}(A)$ is dense in $\mathscr{Q}$: let $h \in \mathscr{Q}$ be such that $\langle u, h \rangle_{\mathscr{Q}} = 0$ for all $u \in \mathcal{D}(A)$. Since A is surjective and h is an element in $\mathscr{H}$, we can find some $v \in \mathcal{D}(A)$ such that $Av = h$. In particular, we have $0 = \langle v, h \rangle_{\mathscr{Q}} = \langle Av, h \rangle = \langle h, h \rangle \implies h = 0$, which proves the density of $\mathcal{D}(A)$ in $\mathscr{Q}$. Since $\mathscr{Q}$ is itself dense in $\mathscr{H}$, A is densely defined in $\mathscr{H}$. Now that we have seen A is a symmetric operator with dense domain and that A is surjective, Proposition I.8 implies that A is self-adjoint. The assertion $q_A(u) = \|u\|_{\mathscr{Q}}^2$ for all $u \in \mathcal{D}(A)$ follows easily from (I.10).

It remains to show the uniqueness. Suppose B is a self-adjoint operator whose domain $\mathcal{D}(B)$ is dense in $\mathscr{Q}$ and whose quadratic form q_B coincides with $\|\cdot\|_{\mathscr{Q}}^2$ on $\mathcal{D}(B)$. Then for all $u, h \in \mathcal{D}(B)$, $\langle Bu, h \rangle_{\mathscr{H}} = q_B(u, h) = \langle u, h \rangle_{\mathscr{Q}}$. Since $\mathcal{D}(B)$ is dense in $\mathscr{Q}$, the last equality remains true for all $h \in \mathscr{Q}$. This says that $(u, Bu) \in \mathcal{G}$, implying $B \subset A$ [3]. For both A and B are self-adjoint, it follows immediately $B \subset A = A^* \subset B^* = B$. ☐

Remark I.14. In fact, we have shown in the above proof that 0 is a resolvent value for the operator A given by (I.10): For any $w \in \mathscr{H}$ there exists a *unique* $u \in \mathcal{D}(A)$ such that $Au = w$. A is therefore bijective from $\mathcal{D}(A)$ to $\mathscr{H}$ and its inverse satisfies the bound $\|A^{-1}\| \leq c^{-1}$.

Theorem I.13 is often used in combination with the following definition

Definition I.15. Let $\mathcal{C} \subset \mathscr{H}$ be a subspace and $A : \mathcal{C} \subset \mathscr{H} \to \mathscr{H}$ an operator defined on $\mathcal{C}$. We say A is **semibounded from below** over $\mathcal{C}$ if there exists $\alpha \in \mathbb{R}$ such that

$$\langle Au, u \rangle \geq \alpha \|u\|^2 , \quad \forall u \in \mathcal{C}.$$

If the operator A is **symmetric**, **semibounded from below** over a **dense subspace** $\mathcal{C}$ in $\mathscr{H}$, then one can always find a sufficiently large constant $a > 0$ such that the quadratic form associated to $A + a$ verifies $q(u) := \langle (A + a)u, u \rangle \geq \|u\|^2$ for all $u \in \mathcal{C}$. Let $\mathscr{Q}$ be the closure of $\mathcal{C}$ by the norm $q(\cdot)^{1/2}$. Then, it is not hard to show that the Hilbert space $(\mathscr{Q}, q(\cdot, \cdot))$ verifies all assumptions of Theorem I.13. Consequently, the theorem ensures us the existence of a domain $\mathcal{D}(A)$, dense in $\mathscr{Q}$ and containing $\mathcal{C}$, on which A is self-adjoint.

[3] This is a shorthand for the statement "A is an extension of B", or $\mathcal{G}(B) \subset \mathcal{G}(A)$.

Example I.16 (Dirichlet Laplacian on $(0,1)$)**.** Consider the one-dimensional Laplacian $A = -\partial_x^2$ on the domain $C_c^\infty((0,1))$. It is obvious that A is semibounded from below. One can take the bilinear form $q_{A+1}(u,v) = \int_0^1 (u'v' + uv)$. The form domain for the operator is then the closure of C_c^∞ with respect to the metric $q_{A+1}^{1/2}$, i.e. $\mathscr{Q} = H_0^1((0,1))$. Since $H_0^1((0,1))$ embeds into $L^2((0,1))$ continuously, Theorem I.13 implies (A, C_c^∞) has a self-adjoint extension $(A, \mathcal{D}(A))$. The domain $\mathcal{D}(A)$ is characterized by (I.10):

$$\mathcal{D}(A) = \left\{ u \in H_0^1((0,1)) : \exists \varphi \in L^2, \forall h \in H_0^1, \quad \int_0^1 u'h' + \int_0^1 uh = \int_0^1 \varphi h \right\}$$

By [Bre10, Theorem IX.25], one sees that every $u \in \mathcal{D}(A)$ has regularity H^2. From which one deduces that $\mathcal{D}(A) = H^2((0,1)) \cap H_0^1((0,1))$.

I.2.3 First example of random operator: Anderson Hamiltonian in dimension 1

With the theory developed in the previous section, we are already able to attack our first example of random Schrödinger operator, the Anderson Hamiltonian in dimension 1, formally represented by

$$\mathcal{H} = -\partial_x^2 + \xi.$$

Here, ξ stands for the Gaussian white noise, a generalized random process that can be regarded as the derivative of Brownian motion in the sense of distributions. More precisely, it is a Gaussian process indexed by the space of test functions $C_c^\infty(\mathbb{R})$, with the covariance function given by $\mathbb{E}[\xi(f)\xi(g)] = \langle f, g \rangle_{L^2}$ for all $f, g \in C_c^\infty$. A way to construct ξ is to pose $\xi(f) = \int f(t)\,\mathrm{d}B(t)$, where $B(t)$ is the one-dimensional Brownian motion and the integral is in Itô's sense.

Let $\mathscr{H} = L^2((0,1))$ be the underlying Hilbert space. The objective of this section is to find a self-adjoint realization of $\mathcal{H}$ on the interval $(0,1)$ satisfying Dirichlet boundary condition. This means that we want to find a (random) domain $\mathcal{D}(\mathcal{H})$ such that a.s. $(\mathcal{H}, \mathcal{D}(\mathcal{H}))$ is self-adjoint and every element $f \in \mathcal{D}(\mathcal{H})$ satisfies $f(0) = f(1) = 0$.

This is not a trivial question as it is not even clear how to choose a suitable $\mathcal{D}(\mathcal{H})$ so that $\mathcal{H}$ takes value in L^2. Indeed, $\mathcal{D}(\mathcal{H})$ can not contain any smooth functions: if $u \in C_c^\infty$, then u'' is still smooth while ξu is only a distribution, leaving no chance for $-u'' + \xi u$ to be in the space L^2.

In this regard, Friedrichs' argument comes in handy, since the bilinear form associated to $\mathcal{H}$ is well-defined on $H_0^1((0,1))$: for all $u, v \in H_0^1((0,1))$,

$$q(u,v) := \int_0^1 u'v' + \int_0^1 u(t)v(t)\,\mathrm{d}B(t) = \int_0^1 u'v' - \int_0^1 (uv)'(t)B(t)\,\mathrm{d}t.$$

Lemma I.17. *Set $M := \sup_{t\in(0,1)} |B(t)|$, and note it is finite a.s.. There exist two constants $c_1 > 0$ and $c_2(M) > 0$ such that almost surely*

$$c_1 \|u\|_{H^1}^2 \le q(u) + 2M^2 \|u\|_{L^2}^2 \le c_2(M) \|u\|_{H^1}^2$$

for all $u \in H_0^1((0,1))$.

Proof. We argue on the almost sure event where M is finite. Observe that for all $\varepsilon > 0$,

$$\left|2\int_0^1 B(t)u(t)u'(t)\,\mathrm{d}t\right| \le \varepsilon^{-1}\int_0^1 |B(t)u(t)|^2 dt + \varepsilon\int_0^1 |u'|^2 \le \frac{M^2}{\varepsilon}\|u\|_{L^2}^2 + \varepsilon\|u'\|_{L^2}^2 .$$

Thus,

$$(1-\varepsilon)\|u'\|_{L^2}^2 \le q(u) + \frac{M^2}{\varepsilon}\|u\|_{L^2}^2 \le (1+\varepsilon)\|u'\|_{L^2}^2 + \frac{2M^2}{\varepsilon}\|u\|_{L^2}^2 .$$

Choose for example $\varepsilon = 1/2$ to conclude the proof. □

Lemma I.17 implies that a.s. the quadratic form q plus a large enough constant is equivalent to the H^1 quadratic form, and therefore the form domain for $\mathcal{H}$ coincides with H_0^1. Since H_0^1 indeed embeds continuously into L^2, Theorem I.13 ensures the existence of a domain $\mathcal{D}(\mathcal{H}) \subset H_0^1$ on which $\mathcal{H}$ is self-adjoint, achieving our objective.

In fact, Lemma I.17 yields even more information. Let $\gamma \ge 2M^2$, one sees that by (I.10), a H_0^1-function u is in the domain $\mathcal{D}(\mathcal{H})$ if and only if there exists some $w \in L^2$ such that $q(u,h) + \gamma\langle u,h\rangle_{L^2} = \langle w,h\rangle_{L^2}$ for any $h \in H_0^1$. That is, u is the weak solution in H_0^1 of the equation $(\mathcal{H}+\gamma)u = w$ for some $w \in L^2$. Friedrichs' theorem and Remark I.14 show that such solution exists and is unique for all $w \in L^2$, and furthermore the mapping $w \mapsto u = (\mathcal{H}+\gamma)^{-1}z$ is bounded from L^2 to H_0^1. This means every $-\gamma$ with $\gamma \ge 2M^2$ is a resolvent value for $\mathcal{H}$, implying $\sigma(\mathcal{H}) \subset (-2M^2, \infty)$. Moreover, as $H_0^1((0,1))$ embeds compactly into $L^2((0,1))$, $(\mathcal{H}+\gamma)^{-1}$ is a compact operator on L^2. This means that the operator $(\mathcal{H}, \mathcal{D}(\mathcal{H}))$ has *compact resolvent*, and therefore its spectrum consists purely of eigenvalues of finite multiplicity.

The same situation also holds for continuous Anderson Hamiltonians on bounded volumes in higher dimensions, as we shall see in the Chapter II.

I.3 RANDOM SCHRÖDINGER OPERATORS IN CONTINUOUS SETTING

I.3.1 Generalities: operator-valued random variables

Having set the playground of self-adjoint operators, we shall now let the probability come into play. In this thesis, as the potential V is random, naturally we would like to see $\mathcal{H} = (i\nabla + \mathbf{A})^2 + V$ as a random variable. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space on which V is defined. One should require

$$\Omega \ni \omega \mapsto \mathcal{H}^\omega = (i\nabla + \mathbf{A})^2 + V^\omega$$

to be measurable. If the map $\mathcal{H}^\omega$ took values in the space $\mathcal{L}(\mathscr{H})$ of bounded operators on the Hilbert space $\mathscr{H}$ (equipped with the norm-topology), then it would simply mean the map should be measurable with respect to the Borel σ-field on $\mathcal{L}(\mathscr{H})$ and to $\mathcal{F}$. However, in the general case, $\mathcal{H}^\omega$ takes values in the space of unbounded operators and it is a priori not clear what topology to put on such space.

The framework to follow here is that of Carmona-Lacroix [CL90].

Definition I.18. Suppose $\mathcal{H}^\omega = (i\nabla + \mathbf{A})^2 + V^\omega$ is an unbounded self-adjoint operator for almost every $\omega \in \Omega$. Let E^ω be the projection-valued measure of $\mathcal{H}^\omega$. We say $\omega \mapsto \mathcal{H}^\omega$ is a *random Schrödinger operator* if one of the three following equivalent conditions holds:

1. $\omega \mapsto E^\omega(B) \in \mathcal{L}(\mathscr{H})$ is measurable for all Borel set $B \in \mathcal{B}(\mathbb{R})$.

2. $\omega \mapsto e^{-it\mathcal{H}^\omega} \in \mathcal{L}(\mathscr{H})$ is measurable for all $t \in \mathbb{R}$.

3. $\omega \mapsto [\mathcal{H}^\omega - z]^{-1} \in \mathcal{L}(\mathscr{H})$ is measurable for all $z \in \mathbb{C}\backslash\mathbb{R}$.

Here, the Banach space $\mathcal{L}(\mathscr{H})$ is equipped with the norm topology and associated Borel σ-field. For a proof of the equivalence, we refer to [CL90, Proposition V.1.2].

An important consequence of Definition I.18 is that many of the spectral quantities associated to a random Schrödinger operator, such as the bottom of spectrum or the eigenvalues, are automatically random variables. This is because they are continuous function functions of the projection-valued measure (in the sense that, if a sequence of projection-valued measures E_n^ω converges to E^ω in norm, then their associated spectra converge as well). This fact is used implicitly in Chapter II.

An important subclass of random operators is that of ergodic ones, which will play an important role in Chapter III. Suppose there exists a family of shift operators $\theta_x : \Omega \to \Omega, x \in \mathbb{R}^d$, such that the random potential $V = V^\omega$ verifies

$$V^{\theta_x\omega}(y) = V^\omega(y - x), \quad \forall y \in \mathbb{R}^d, \omega \in \Omega.$$

We also assume that θ_x is a measure-preserving automorphism on Ω. That is, θ_x is bijective for all $x \in \mathbb{R}^d$ such that both θ_x and θ_x^{-1} are measurable, and that $\mathbb{P}(\theta_x A) = \mathbb{P}(A)$ for all $A \in \mathcal{F}$. We can now recall the definition of ergodicity.

Definition I.19. The family $(\theta_x)_{x\in\mathbb{R}^d}$ is said to be *ergodic* for the law $\mathbb{P}$ of V, if every event $A \in \mathcal{F}$ such that $\mathbb{P}((\theta_x^{-1}A)\Delta A) = 0$ has probability $\mathbb{P}(A)$ either 0 or 1.

To state the ergodic property for random operators, we need a representation of (θ_x) in terms of operators on $\mathscr{H}$. This is done by defining the family $(\mathcal{T}_x)_{x\in\mathbb{R}^d}$ through

$$\mathcal{T}_x f(\cdot) = f(\cdot + x), \quad f \in \mathscr{H}.$$

It is clear that $\mathcal{T}_x$ is unitary on $\mathscr{H}$, with its adjoint given by

$$\mathcal{T}_x^* f(\cdot) = f(\cdot - x), \quad f \in \mathscr{H}.$$

Definition I.20. Given that the family (θ_x) is ergodic for $\mathbb{P}$, we say that a random self-adjoint operator $\mathcal{H} = \mathcal{H}^\omega$ is ergodic if it holds

$$\mathcal{H}^{\theta_x\omega} = \mathcal{T}_x^* \mathcal{H}^\omega \mathcal{T}_x, \quad \forall \omega \in \Omega, x \in \mathbb{R}^d$$

An important consequence of the ergodicity for random operators is that it implies the spectrum is deterministic (a.s.).

Proposition I.21. *If the random self-adjoint operator $\omega \mapsto \mathcal{H}^\omega$ is ergodic in the sense of Definition I.20, then there exists a closed set $\Sigma \subset \mathbb{R}$ such that for $\mathbb{P}$-almost every $\omega \in \Omega$ one has*

$$\sigma(\mathcal{H}^\omega) = \Sigma.$$

Proof. Let (e_n) be an orthonormal basis for $\mathscr{H}$. Let E^ω be the projection-valued measure of $\mathcal{H}^\omega$. Uniqueness of spectral theorem implies that $E^{\theta_x\omega}(B) = \mathcal{T}_x^* E^\omega(B)\mathcal{T}_x$ for all $B \in \mathcal{B}(\mathbb{R})$. One thus deduces

$$\operatorname{tr} E^{\theta_x\omega}(B) = \sum_{n\geq 1} \langle E^\omega(B)\mathcal{T}_x e_n, \mathcal{T}_x e_n\rangle = \operatorname{tr} E^\omega(B), \quad \forall \omega \in \Omega, x \in \mathbb{R}^d, B \in \mathcal{B}(\mathbb{R}^d).$$

This shows that $\omega \mapsto \operatorname{tr} E^\omega(B)$ is (θ_x)-invariant. By ergodicity, it follows that for every $B \in \mathcal{B}(\mathbb{R})$, $\operatorname{tr} E^\omega(B) = r_B$ a.s. for some $r_B \in [0, \infty]$. Now one can define

$$\Sigma = \Big\{\lambda \in \mathbb{R} : r_{(\lambda_1,\lambda_2)} \neq 0, \quad \forall \lambda_1, \lambda_2 \in \mathbb{Q} \text{ such that } \lambda_1 < \lambda < \lambda_2\Big\}$$
$$\Omega_0 = \bigcap_{\lambda_1,\lambda_2\in\mathbb{Q}:\lambda_1<\lambda_2} \Big\{\omega \in \Omega : \operatorname{tr} E^\omega((\lambda_1, \lambda_2)) = r_{(\lambda_1,\lambda_2)}.\Big\}$$

It is then easy to see that Ω_0 is of probability one and that for every $\omega \in \Omega_0$, E^ω has topological support Σ. □

I.3.2 Construction

Having introduced the notion of random Schrödinger operator, in the following we will examine how to construct such objects starting from a formal expression $\mathcal{H} = \mathcal{H}_0 + V$. Throughout this section, we will consider the dimensions $d \in \{2, 3\}$ and $\mathcal{H}_0$ is taken to be the magnetic Laplacian

$$\mathcal{H}_0 = (i\nabla + \mathbf{A})^2$$

with $\mathbf{A}$ given by (I.4) or (I.5). As our potential V will be random, the objective is to identify a random dense domain $\mathcal{D}(\mathcal{H})$ on which $\mathcal{H}$ is well-defined and self-adjoint; moreover, we want $\mathcal{H}$ to be measurable as a random variable in the sense of Definition I.18. This task is not always trivial: as we have already seen in section I.2.3, the one-dimensional continuous Anderson Hamiltonian cannot contain any smooth functions in its domain and its construction is not done in a classical way – this operator is not obtained by an extension of a symmetric operator over C_c^∞.

We will propose three ideas of construction based on the two equations (I.11) and (I.12) below. Each of these ideas will be illustrated in the simple setting of continuous bounded potential. These approaches will serve as the basic guideline and can be generalized to the case of rough noise subsequently.

Our guiding idea is to transform the problem of construction to the solution theory of related SPDEs. Consider the elliptic equation

$$(\mathcal{H} + a)u = g \tag{I.11}$$

with given data $g \in L^2$ and $a \in \mathbb{R}$. Suppose that one manages to solve (I.11) for a unique u with any given $g \in L^2$ and shows that the map $R_a : g \mapsto u$ is a bounded operator. Heuristically, this is somehow equivalent to construct the inverse of the not yet defined operator $\mathcal{H} + a$. Indeed, one sees clearly that R_a must be injective and therefore is bijective from L^2 to its range. It is thus tempting to define $\mathcal{H}$ as $R_a^{-1} - a$. In fact, R_a can actually be interpreted as the resolvent $(\mathcal{H} + a)^{-1}$: this is done by showing an algebraic relation, called the *resolvent identity*, satisfied by the family of operators (R_a) indexed by values of a for which R_a is defined. The resolvent identity ensures that the unbounded operator $(R_a^{-1} - a, \mathrm{Ran}(R_a))$ is independent of a and therefore can be taken as a rigorous definition of $\mathcal{H}$. Under this construction, $\mathcal{H}$ comes with a natural domain $\mathcal{D}(\mathcal{H}) = \mathrm{Ran}(R_a)$ and it is usually not hard to show that $\mathcal{H}$ is symmetric on $\mathcal{D}(\mathcal{H})$. As $\mathcal{H}$ admits a resolvent value $-a$ by construction, Proposition I.8 implies $(\mathcal{H}, \mathcal{D}(\mathcal{H}))$ is self-adjoint. We will implement this argument with an example further below.

Instead of (I.11), one can also consider the formal parabolic equation:

$$(\partial_t + \mathcal{H})u = 0. \tag{I.12}$$

Suppose one solves (I.12) with initial data u_0 living in a dense subspace of $\mathscr{H}$ and define the linear map $R_t : u_0 \mapsto u(t, \cdot)$. Provided that R_t extends continuously as a bounded operator on $\mathscr{H}$ and that (R_t) forms a *strongly continuous semigroup* of self-adjoint bounded operators, that is, (R_t) verifies

1. For each $t > 0$, R_t is a bounded operator on $\mathscr{H}$. Also, R_t is symmetric, i.e. $\langle R_t f, g\rangle_{\mathscr{H}} = \langle f, R_t g\rangle_{\mathscr{H}}$.

2. (Semigroup) For all $t, s > 0$. $R_{t+s} = R_t \circ R_s$.

3. (Strongly continuous) For each $f \in \mathscr{H}$, $\lim_{t\to 0} \|R_t f - f\|_{L^2} = 0$,

then classical Hille-Yosida theory (see for instance [Rud91, Theorem 13.35]) promises the existence of a unique self-adjoint generator $(\mathcal{H}, \mathcal{D}(\mathcal{H}))$ of $(R_t)_{t>0}$. More precisely, one poses

$$\begin{aligned} \mathcal{D}(\mathcal{H}) &= \Big\{ f \in \mathscr{H} : \text{ The limit of } t^{-1}(R_t f - f) \text{ exists in } \mathscr{H} \text{ when } t \to 0 \Big\} \\ \mathcal{H} f &= \lim_{t\to 0} t^{-1}(R_t f - f), \quad f \in \mathcal{D}(\mathcal{H}) \end{aligned}$$

The domain $\mathcal{D}(\mathcal{H})$ is indeed dense in $\mathscr{H}$ since for every $f \in \mathscr{H}$, the element $f_t := t^{-1} \int_0^t R_s f \,\mathrm{d}s$ is in the domain $\mathcal{D}(\mathcal{H})$ and one has $f_t \to f$ in $\mathscr{H}$ thanks to the continuity of $s \mapsto R_s f$. The symmetry of $\mathcal{H}$ on $\mathcal{D}(\mathcal{H})$ is a direct consequence of the symmetry of R_t on $\mathscr{H}$ and the continuity of the $\mathscr{H}$-scalar product. Moreover, by the semigroup property and the strong continuity of R_t, one can show that there exist constants $C, a > 0$ such that $\|R_t\| \le Ce^{at}$. Consequently, for every $\lambda > a$, one can deduce the integral $\int_0^\infty e^{-\lambda t} R_t f \,\mathrm{d}t$ converges for every $f \in \mathscr{H}$ and coincides with the resolvent $(\mathcal{H} + \lambda)^{-1} f$. Therefore, $\mathcal{H}$ is a symmetric operator on a dense domain and every $-\lambda$ with $\lambda > a$

is a resolvent value for $\mathcal{H}$, implying $\mathcal{H}$ is self-adjoint (Proposition I.8). As a corollary, the spectrum of $\mathcal{H}$ verifies $\sigma(\mathcal{H}) \subset (-a, \infty)$. This provides us another alternative to construct $\mathcal{H}$.

In what follows, we will solve explicitly (I.11) and (I.12) in a relatively simple setting: we concentrate on the case where V is continuous and bounded. Let $\mathscr{B}$ denote the Banach space $\mathscr{B} = C(\mathbb{T}^d; \mathbb{R})$ of real-valued continuous functions on the torus $\mathbb{T}^d$ (i.e., continuous on $[0,1)^d$ and is $\mathbb{Z}^d$-periodic) equipped with the usual supremum norm $\|\cdot\|_\infty$. Let the potential V be a random variable taking values in $\mathscr{B}$. That is, the mapping

$$V : \omega \mapsto V^\omega \in \mathscr{B}$$

defines a measurable function from $(\Omega, \mathcal{F})$ to $(\mathscr{B}, \|\cdot\|_\infty)$.

Let us point out that there are two main approaches which we use to solve (I.11): the mild and the weak solution, where the former is closely related to the formulation in regularity structures and thus will serve as the central example in section I.4, while the latter resorts to Friedrichs' extension studied in section I.2.2.

We shall prove the following theorem using three different methods.

Theorem I.22. *In dimensions $d = 2, 3$, there exists some random domain $\mathcal{D}(\mathcal{H}) \subset L^2([0,1]^d)$ with periodic boundary condition such that the operator $\mathcal{H} = \mathcal{H}_0 + V$ is self-adjoint on $\mathcal{D}(\mathcal{H})$. Moreover, $(\mathcal{H}, \mathcal{D}(\mathcal{H}))$ is a random Schrödinger operator in the sense of Definition I.18.*

Construction I: mild solution

Fix $a > 0$. Let us first collect some properties of the Green function $K_a : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ for $(\mathcal{H}_0 + a)^{-1}$ on $\mathbb{R}^d$, i.e., the solution in the sense of distribution to the equation

$$[(i\nabla + \mathbf{A})^2 + a] K_a(x, y) = \delta_x(y)$$

(where the differential operator acts on the variable y). Pose

$$K_a(x, y) = e^{i\frac{B}{2}(x_1 y_2 - x_2 y_1)} G(x, y), \quad x, y \in \mathbb{R}^d.$$

With direct computation, one can show that G obeys the equation

$$(-\Delta + a) G(x, y) = e^{-i\frac{B}{2}(x_1 y_2 - x_2 y_1)} \delta_x(y).$$

With the condition that $G(x, y) \to 0$ as $|x - y| \to \infty$, there exists a unique solution G and it can be checked [Uet92] that G is a complex-valued oscillatory function continuous on $\mathbb{R}^d \times \mathbb{R}^d \backslash \{(x, y) : x = y\}$ whose absolute value behaves like the Green function for $-\Delta + a$ when $|x - y| \to 0$. That is

$$|K_a(x, y)| = |G(x, y)| \sim \begin{cases} -\frac{1}{2\pi} \log |x - y|, & d = 2, \\ \frac{1}{4\pi |x - y|}, & d = 3, \end{cases} \quad \text{as } |x - y| \to 0. \tag{I.13}$$

Moreover,

$$|K_a(x,y)| \le e^{-\sqrt{a}|x-y|}, \quad \text{for } |x-y| \text{ large enough.} \tag{I.14}$$

For any function $f \in L^2(\mathbb{T}^d)$, define the function $K_a f$ by $K_a f(x) = \int_{\mathbb{R}^d} K_a(x,y) f(y)\,\mathrm{d}y$. A priori, the function $K_a f$ will not be $\mathbb{Z}^d$-periodic, whence not a funtion defined on $\mathbb{T}^d$. However, we can periodize:

$$\tilde{K}_a f(x) = \begin{cases} K_a f(x), & x \in [0,1)^d \\ K_a f(x-k), & x_j \in [k_j, k_j+1),\ k = (k_1, \dots, k_d) \in \mathbb{Z}^d. \end{cases}$$

Consequently, the function $\tilde{K}_a f \in L^2(\mathbb{T}^d)$ for every $f \in L^2(\mathbb{T}^d)$ and satisfies $[(i\nabla + \mathbf{A})^2 + a]\tilde{K}_a f = f$ on $\mathbb{T}^d$ in distributional sense; furthermore, by (I.13), (I.14) we have the estimate

$$\left\| \tilde{K}_a f \right\|_{L^2(\mathbb{T}^d)} \le C a^{-1} \|f\|_{L^2(\mathbb{T}^d)} \tag{I.15}$$

for all $f \in L^2(\mathbb{T}^d)$ with a uniform constant C.

The mild formulation consists in writing (I.11) into the form:

$$u = \tilde{K}_a(g - Vu). \tag{I.16}$$

Define the linear map

$$\mathcal{M}_a : u \mapsto \tilde{K}_a(g - Vu).$$

The aim is to show that the map $\mathcal{M}_a$ is a well-defined linear map on $L^2(\mathbb{T}^d)$ and admits a fixed point for $a > 0$ sufficiently large. Indeed, (I.15) implies $\mathcal{M}_a u$ is in L^2 for every $u \in L^2$; moreover, given any two elements $u_1, u_2 \in L^2(\mathbb{T}^d)$, one can compute

$$\|\mathcal{M}_a u_1 - \mathcal{M}_a u_2\|_{L^2} \le C a^{-1} \|V(u_1 - u_2)\|_{L^2} \le C a^{-1} \|V\|_{\mathscr{B}} \|u_1 - u_2\|_{L^2}.$$

Hence, given $g \in L^2$ and any positive real a larger than the random value $C\|V\|_{\mathscr{B}}$, the map $\mathcal{M}_a$ admits a unique fixed point u. Let us show that the fixed point u depends continuously on g and on V. Suppose $g_1, g_2 \in L^2$, $V_1, V_2 \in \mathscr{B}$ and $a > C(\|V_1\|_{\mathscr{B}} + \|V_2\|_{\mathscr{B}})$. Let u_1, u_2 be such that $u_j = K_a(g_j - V_j u_j)$ for $j = 1, 2$. A computation shows

$$\|u_1 - u_2\|_{L^2} \le C a^{-1} \left(\|g_1 - g_2\|_{L^2} + \|V_1 - V_2\|_{\mathscr{B}} \|u_1\|_{L^2} + \|V_2\|_{\mathscr{B}} \|u_1 - u_2\|_{L^2} \right)$$

or

$$\|u_1 - u_2\|_{L^2} \le \frac{C a^{-1}}{1 - C a^{-1} \|V_2\|_{\mathscr{B}}} \left(\|g_1 - g_2\|_{L^2} + \|V_1 - V_2\|_{\mathscr{B}} \|u_1\|_{L^2} \right). \tag{I.17}$$

Now, define the map R_a which associates to every $g \in L^2$ the unique fixed point u. From the above estimates one deduces that for any fixed realization of potential $V \in \mathscr{B}$ and $a > C\|V\|_{\mathscr{B}}$, R_a is a bounded linear operator on L^2 with operator-norm bounded by $\frac{Ca^{-1}}{1 - Ca^{-1}\|V\|_{\mathscr{B}}}$. Furthermore, it follows from (I.17) that the map $V \mapsto R_a$ is continuous from $\mathscr{B}$ to $\mathcal{L}(L^2)$.

So far, we have proved that R_a is a bounded operator on $L^2(\mathbb{T}^d)$ provided that the parameter $a > 0$ is large (than a random number depending on V). Now it remains to show, with a fixed realization $V \in \mathscr{B}$, R_a is in fact the resolvent of some self-adjoint operator. To do this, we first remark that R_a is injective: Suppose g is some L^2-function with the property $R_a g = 0$. Then one must have $\tilde{K}_a g = 0$, which implies $g = 0$ (since $-a$ is a resolvent value for $(i\nabla + \mathbf{A})^2$). Moreover, the image of R_a does not depend on a: it is a consequence of the following identity which can be verified by direct calculation

$$R_{a_1} - R_{a_2} = (a_2 - a_1) R_{a_2} R_{a_1}.$$

Therefore, one can define the the unbounded operator $\mathcal{H} = R_a^{-1} - a$ on the domain $\mathcal{D}(\mathcal{H}) = \operatorname{Ran} R_a$. This operator $(\mathcal{H}, \mathcal{D}(\mathcal{H}))$ is independent of the parameter a. Actually, the domain $\mathcal{D}(\mathcal{H})$ coincides with $H^2(\mathbb{T}^d)$ which is dense in L^2: for one inclusion, let $u \in H^2(\mathbb{T}^d)$ and we see that $g = (\mathcal{H}_0 + V + a)u$ is in $L^2(\mathbb{T}^d)$, implying $u = R_a g \in \operatorname{Ran} R_a$; conversely, by the fixed point relation, the image $\operatorname{Ran} R_a$ is included into the domain of the reference operator $\mathcal{H}_0 = (i\nabla + \mathbf{A})^2$, which is nothing but $H^2(\mathbb{T}^d)$.

Finally, we show that $(\mathcal{H}, \mathcal{D}(\mathcal{H}))$ is self-adjoint. Note that $\mathcal{H}$ is symmetric over $\mathcal{D}(\mathcal{H})$: this is because V is real-valued and that the space $(i\nabla + \mathbf{A})^2$ is symmetric on $H^2(\mathbb{T}^d)$. Since $\mathcal{H}$ is symmetric over $\mathcal{D}(\mathcal{H})$ and $\mathcal{H} + a$ is surjective for all a large enough, it follows from Proposition I.8 that $\mathcal{H}$ is self-adjoint. By construction, its resolvent coincides with R_a.

Since we have already seen $V \mapsto R_a = (\mathcal{H}_0 + V + a)^{-1}$ is continuous, combined with the assumption that the $\omega \to V^\omega$ is measurable we conclude that $\mathcal{H} : \omega \mapsto \mathcal{H}_0 + V^\omega$ satisfies the property 3 of Definition I.18 and hence is indeed a random Schrödinger operator.

Construction II: weak solution - Friedrichs

The weak solution, on the other hand, considers the weak formulation of (I.11): For any given $g \in L^2$, we want to find a suitable function space $\mathscr{Q}$ and $u \in \mathscr{Q}$ such that

$$\langle (\mathcal{H}_0 + V)u, h \rangle + a \langle u, h \rangle = \langle g, h \rangle, \quad \forall h \in \mathscr{Q} \tag{I.18}$$

A comparison of (I.18) to (I.10) shows that this formulation is equivalent to finding the form domain $\mathscr{Q}$ for the formal bilinear form $q_a(u, v) = \langle (\mathcal{H}_0 + V)u, v \rangle + a \langle u, v \rangle$ and realize $\mathcal{H}$ as the self-adjoint operator given by Friedrichs' theorem I.13.

Since it is known that the form domain for the operator $\mathcal{H}_0$ on $\mathbb{T}^d$ is the Sobolev space $H^1(\mathbb{T}^d)$, let us take $\mathscr{Q} = H^1(\mathbb{T}^d)$ and see if the form q_a is well-defined and closed on $\mathscr{Q}$. Indeed, for all $u \in L^2$ we can estimate

$$| \langle Vu, u \rangle | \leq \|V\|_{\mathscr{B}} \|u\|_{L^2}^2 .$$

This entails that

$$(a - \|V\|_{\mathscr{B}}) \|u\|_{L^2}^2 \leq q_a(u, u) - \langle \mathcal{H}_0 u, u \rangle_{L^2} \leq (a + \|V\|_{\mathscr{B}}) \|u\|_{L^2}^2$$

for every $u \in \mathscr{Q}$. Therefore, provided $a > \|V\|_{\mathscr{B}}$, the norm $\|u\|_{\mathscr{Q}} = q_a(u, u)^{1/2}$ is equivalent to the H^1-norm and hence is closed on $\mathscr{Q}$. This means that $(\mathscr{Q}, q_a)$ is a Hilbert space verifying the assumption of Theorem I.13.

Consequently, we can define $\mathcal{H}$ as the self-adjoint operator given by (I.10). More explicitly, the domain of $\mathcal{H}$ coincides with the family of weak solutions to (I.18) in $\mathscr{Q}$:

$$\mathcal{D}(\mathcal{H}) = \left\{ u \in H^1(\mathbb{T}^d) : \exists g \in L^2, a > \|V\|_{\mathscr{B}}, \forall h \in H^1,\ q_a(u,h) = \langle g, h\rangle_{L^2} \right\}$$

and that for every $u \in \mathcal{D}(\mathcal{H})$, $(\mathcal{H}+a)u = g$ with g given in the above equation.

It is left to show that $\mathcal{H}$ satisfies Definition I.18. We will do this by showing the resolvent $(\mathcal{H}+a)^{-1}$ depends continuously on the realization of potential V. Suppose $g \in L^2$ and that we have a sequence (V_n) converging to V in $\mathscr{B}$. Let $u_n, u \in \mathscr{Q}$ and $a > \sup_n \|V_n\|_{\mathscr{Q}}$ be such that $\langle (\mathcal{H}_0 + V_n)u_n, h\rangle + a\langle u_n, h\rangle = \langle g, h\rangle$ for all $n \geq 0$ and $h \in \mathscr{Q}$. A computation shows

$$\langle (\mathcal{H}_0 + V + a)(u_n - u), h\rangle = \langle (V - V_n)u_n, h\rangle$$

for all $h \in H^1_0$. By taking $h = u_n - u$, one has in particular

$$(a - \|V\|_{\mathscr{B}})\, \|u_n - u\|_{L^2} \leq \|V - V_n\|_{\mathscr{B}}\, \|u_n\|\,.$$

Since we have $\sup_n \|u_n\|_{L^2} \leq \frac{\|g\|_{L^2}}{a - \sup_n \|V_n\|_{\mathscr{B}}} < \infty$ and that V_n converges to V, one conclude $u_n \to u$ in L^2. In fact, we have proved the following: If $V_n \to V$ in $\mathscr{B}$, then $\mathcal{H}_0 + V_n$ converges to $\mathcal{H}_0 + V$ in norm-resolvent sense. As $\omega \mapsto V^\omega$ is measurable, this completes the proof.

Construction III: Schrödinger semigroup

Let us now implement the argument outlined previously based on the equation (I.12). We consider the Cauchy problem to the PDE

$$\begin{cases} \partial_t u = -(\frac{1}{2}\mathcal{H}_0 + V)u, & \text{in } \mathbb{R}_+ \times \mathbb{T}^d, \\ u(0,\cdot) = u_0(\cdot), \end{cases} \tag{I.19}$$

for a given initial data $u_0 \in C_c^\infty(\mathbb{T}^d)$. For notational simplicity, we include a prefactor $1/2$ in front of the operator $\mathcal{H}_0$. It turns out that the PDE admits a unique solution, denoted by $u = u(t,x)$, which we will describe later. Write $u(t) = u(t,\cdot)$ and define the linear map $R_t u_0 = u(t)$ as well as $R_0 = I$. The objective is to verify

1. For each $t > 0$, R_t is a bounded operator on $L^2(\mathbb{T}^d)$. Also, R_t is symmetric, i.e. $\langle R_t f, g\rangle_{L^2} = \langle f, R_t g\rangle_{L^2}$.
2. (Semigroup) For all $t, s > 0$. $R_{t+s} = R_t \circ R_s$.
3. (Strongly continuous) For each $f \in L^2(\mathbb{T}^d)$, $\lim_{t\to 0} \|R_t f - f\|_{L^2} = 0$.

If these three properies are true, Hille-Yosida theorem yields a unique, self-adjoint generator $\mathcal{H}$ for the strongly continuous symmetric semigroup (R_t). This operator $\mathcal{H}$ can then be taken as the definition of $\frac{1}{2}\mathcal{H}_0 + V$.

To find a soluton to the PDE, one can choose to take the mild formulation just as in the elliptic case, i.e. find the fixed point u of the following equation in a certain space of space-time functions:

$$u(t) = P_t u_0 - \int_0^t P_s(Vu)\,\mathrm{d}s$$

where (P_t) denotes the (magnetic) heat kernel $(\partial_t + \mathcal{H}_0)^{-1}$. Using the a priori estimate of P_t, one can play with the parameter t and show that the fixed point exists and is unique up to a sufficiently short time horizon $T > 0$.

We choose not to do so, but rather to provide a probabilitic argument via the Feyman-Kac-Itô formula. Introduce a d-dimensional Brownian motion (B_t) living in $C(\mathbb{R}_+; \mathbb{T}^d)$ with its law given by the probability measure P. From now on, we should work on the product probability space $\Omega \times C(\mathbb{R}_+; \mathbb{T}^d)$ equipped with the product Borel σ-algebra and the product measure $\mathbb{P} \otimes P$ (see [Hsu02] for a presentation of Brownian motion on a manifold). Denote E_x the expectation with respect to the conditional measure $P(\cdot | B_0 = x)$. For $t > 0, x \in \mathbb{T}^d$, define

$$R_t^V f(x) = E_x\left[f(B_t)e^{-F_t}\right] \tag{I.20}$$

where

$$F_t = i\int_0^t \mathbf{A}(B_s)\cdot \mathrm{d}B_s + \frac{i}{2}\int_0^t \nabla\cdot\mathbf{A}(B_s)\,\mathrm{d}s + \int_0^t V(B_s)\,\mathrm{d}s. \tag{I.21}$$

Here the integral $\int_0^t \mathbf{A}(B_s)\cdot\mathrm{d}B_s$ is interpreted in Itô's sense.

Let us first check that (R_t^V) verifies the properties 1.-3. and therefore admits a self-adjoint generator $(\mathcal{H}, \mathcal{D}(\mathcal{H}))$. Indeed,

1. The symmetry of R_t^V follows from the time-reversal property, i.e. $(B_s)_{0<s<t} = (B_{t-s})_{0<s<t}$ in law. The boundedness follows from the estimate

$$\left\|R_t^V f\right\|_{L^2}^2 \le e^{2t\|V\|_{\mathscr{B}}} \int_{\mathbb{T}^d} E_x[|f(B_t)|^2]\,\mathrm{d}x \le (e^{t\|V\|_{\mathscr{B}}}\|f\|_{L^2})^2$$

   where in the second inequality we used the fact that the transition density for (B_t) is bounded.

2. The semigroup identity is a consequence of the Markov property of the Brownian motion.

3. Since continuous functions are dense in L^2 and that R_t^V is bounded, it suffices to show that $\left\|R_t^V f - f\right\|_{L^2} \to 0$ for all $f \in C(\mathbb{T}^d)$. Note that

$$|R_t^V f(x) - f(x)| \le E_x\left|f(B_t)e^{-F_t} - f(B_0)\right|.$$

   For all $x \in \mathbb{T}^d$, it follows from the continuity of f and Brownian motion that $f(B_t)e^{-F_t}$ converges to $f(B_0)$ a.s.. Moreover, since one has $|f(B_t)e^{-F_t} - f(B_0)| \le \|f\|_{L^\infty([0,1]^d)}(e^{t\|V\|_{\mathscr{B}}} + 1)$, the dominated convergence theorem (w.r.t. the expectation E_x) implies that $R_t^V f(x) - f(x) \to 0$ pointwise in $\mathbb{T}^d$. Another application of dominated convergence (w.r.t. the Lebesgue measure on $\mathbb{T}^d$) then implies $\left\|R_t^V f - f\right\|_{L^2} \to 0$.

To see that this generator $\mathcal{H}$ serve as a self-adjoint realization for the formal operator $\frac{1}{2}\mathcal{H}_0 + V$, we show that R_t^V does solve (I.19). Let $f \in C_c^\infty(\mathbb{T}^d)$. Itô's formula yields

$$R_t^V f(x) = f(x) - \int_0^t R_s^V \left(\frac{1}{2}\mathcal{H}_0 + V\right) f(x)\,\mathrm{d}s, \quad x \in \mathbb{T}^d.$$

It can be checked that, for fixed $x \in \mathbb{T}^d$, the function $s \mapsto R_s^V(\frac{1}{2}\mathcal{H}_0 + V)f(x)$ is continuous. This implies the function $u(t,x) = R_t^V f(x)$ is differentiable in time and solves the equation (I.12) pointwise with initial condition $f \in C_c^\infty$.

Finally, the continuity of R^V with respect to V follows from the estimate

$$\begin{aligned}\left\|(R_t^V - R_t^{V_n})f\right\|_{L^2}^2 &\leq \int_{\mathbb{T}^d} E_x[|f(B_t)|^2] \cdot e^{2t\|V\|_{\mathscr{B}}} E_x\left[\left|1 - e^{\int_0^t (V-V_n)(B_s)\,\mathrm{d}s}\right|^2\right] \mathrm{d}x \\ &\leq (Ct\,\|V - V_n\|_{\mathscr{B}}\, e^{t(\|V\|_{\mathscr{B}} + \|V-V_n\|_{\mathscr{B}})}\, \|f\|_{L^2})^2.\end{aligned}$$

It follows that the map $\omega \mapsto R_t^{V^\omega}$ is a measurable function from Ω to $\mathcal{L}(\mathscr{H})$. As R_t^V coincides with $e^{-t(\frac{1}{2}\mathcal{H}_0 + V)}$, one concludes that $\mathcal{H}$ is indeed a random Schrödinger operator.

I.3.3 The full space case

Before inspecting the case with rough random noise, let us now discuss the situation where the potential V remains continuous but is defined on the full Euclidean space $\mathbb{R}^d$. Indeed, even for continuous $V \in C(\mathbb{R}^d;\mathbb{R})$, the arguments proposed in section I.3.2 will in general fail due to the unboundedness of V. This implies that we probably need assumptions on the growth of the potential V at infinity.

It turns out that the *negative growth* of V plays an important role when it comes to self-adjointness. For simplicity, suppose $V(x) = -q(|x|)$, $x \in \mathbb{R}^d$, for some increasing function $q : \mathbb{R}_+ \to \mathbb{R}_+$. The following heuristic from physics gives a good idea on how should the growth of q behave in order for the operator $-\Delta + V$ to be essentially self-adjoint (over C_c^∞): Consider a particle subject to the potential V. Under this potential, one expects the particle to accelerate in the radial direction. In physics, the interpretation for the essential self-adjointness is that the dynamics of the particle should be completely determined by its Hamiltonian operator and thus should not *escape to infinity* (otherwise we would need extra *boundary condition* to specify the behavior of the particle after explosion). We can study this situation using an intuition from classical mechanics: according to the conservation of energy, the speed of the particle at position $x \in \mathbb{R}^d$ should be given by $\sqrt{C - V(x)}$ for some constant C. Consequently, given the initial position x_0 of the particle, the time needed for it to flee to infinity can be computed by the integral $\int_{|x_0|}^\infty \frac{\mathrm{d}r}{\sqrt{C+q(r)}}$. This quantity of time is infinite, and hence the operator is essentially self-adjoint, if the function q satisfies

$$\int_{|x_0|}^\infty q(r)^{-1/2}\,\mathrm{d}r = \infty \tag{I.22}$$

for any $x_0 \in \mathbb{R}^d$. In particular, if the potential takes the form $V(x) = -|x|^\alpha$, then (I.22) implies α should be such that $\alpha \leq 2$.

Kato [Kat72] confirmed this physical intuition by proving the following result: the operator $-\Delta + V$ is essentially self-adjoint, provided that $V = V_1 + V_2$, where $V_1 \in L^2_{\text{loc}}$ is such that $V_1(x) \geq -q(|x|)$ with a increasing function q such that $q(r) = o(r^2)$ as $r \to \infty$ (which indeed verifies (I.22)) and V_2 belongs to the so called *Kato class* K^d. Morally speaking, the Kato class function V_2 is such that it is $(-\Delta)-$bounded with relative bound 0: this means that V_2 can be seen as a perturbation to $-\Delta$ preserving the self-adjointness. On the other hand, the constraint on V_1 says that we can allow the potential to grow at any rate to $+\infty$ (as long as it remains L^2_{loc}), but the divergence to $-\infty$ has to be limited by $-|x|^{2-\varepsilon}$.

We refrain from giving a complete definition of the Kato class K^d but refer the readers to the original work of Kato [Kat72] and the surveys by Simon [Sim82], [Sim18]. However, we do mention that there is a close connection between the Kato class and the Brownian motion: the local Kato class K^d_{loc} coincides with the largest family of potentials V for which we can have a Feyman-Kac representation for the Schrödinger semigroup $e^{-t(-\Delta+V)}$, see [AS82], [Sim82] and [Szn98] for details. In this regard, the idea proposed in section I.3.2, Construction III, could be generalized to cover operators with random potentials taking values in K^d_{loc}.

We remark that the principal source of noise we will consider in the sequel, the Gaussian white noise ξ (Definition I.23), exhibits a logarithmic divergence at infinity, hence satisfies (I.22). (Morally speaking, the minimum of white noise ξ on the cube $(-L/2, L/2)^d$ can be approximated by the minimum of a family of $O(L^d)$ i.i.d. Gaussian random variables – this quantity is of order $-\sqrt{\log(L^d)}$.) Therefore, we do expect that $\mathcal{H} = (i\nabla + \mathbf{A})^2 + \xi$ can be defined on $L^2(\mathbb{R}^d)$, provided that we solve the singularity issue which will be treated in the next section.

Indeed, Ugurcan [Ugu22] proposed the following idea to prove the self-adjointness of the Anderson Hamiltonian $-\Delta + \xi$ on $L^2(\mathbb{R}^2)$ via a commutator theorem due to Faris and Lavine (Proposition IV.16) which we summarized below:

> For a symmetric operator $(\mathcal{H}, \mathcal{D}(\mathcal{H}))$, if there exists a positive, self-adjoint operator $(N, \mathcal{D}(N))$ such that $\mathcal{D}(N) \subset \mathcal{D}(\mathcal{H})$ and
>
> $$|\langle Nu, \mathcal{H}u\rangle - \langle \mathcal{H}u, Nu\rangle| \lesssim \langle Nu, u\rangle\,, \quad \forall u \in \mathcal{D}(N),$$
>
> then $\mathcal{H}$ is essentially self-adjoint on $\mathcal{D}(\mathcal{H})$.

In light of this tool, Ugurcan's idea is to decompose the white noise ξ into a singular part ξ_-, which is a globally Hölder distribution (thus rough but "bounded"), and an unbounded part ξ_+, which is a function satisfying the condition $\xi_+(x) \geq -C|x|^2$; he then defines the operator $-\Delta + \xi_-$ using singular SPDE techniques and argues that $\mathcal{H} = -\Delta + \xi_- + \xi_+$ is essentially self-adjoint using Faris-Lavine with the auxilary operator $N = \mathcal{H} + c(1 + |x|^2)$. Inspired by this reasoning, we try to extend his argument to magnetic case in Chapter IV, an effort remaining however unsuccessful by the completion of the present thesis. The reason is that our proof for the self-adjointness of N is incomplete, and there is already a similar gap present in [Ugu22], see the discussion in section I.5 and in the begining of Chapter IV.

Let us point out that we present in this thesis an alternative to Ugurcan's argument. This approach relies on a theorem due to Klein and Landau (Theorem III.3) concerning the semigroup formed by symmetric unbounded operators, which can be seen as an extension of the Schrödinger semigroup idea presented in section I.3.2. This idea is implemented in detail in Chapter III. We believe the same approach can also be applied to the magnetic case considered in Chapter IV.

I.3.4 The rough case and the state of the art

In the previous discussions, the construction of random operators can be rephrased as PDE problems (I.11) and (I.12). However, there are situations where (I.11) and (I.12) are classically ill-posed. This happens when the random potential V has too low regularity so that the product term Vu appearing in (I.11) and (I.12) may not even be defined.

The most important example for such potential V is the spatial Gaussian white noise.

Definition I.23 (Gaussian white noise)**.** The spatial Gaussian white noise is the centered Gaussian field $\xi = (\xi(\varphi))_{\varphi \in C_c^\infty(\mathbb{R}^d)}$ indexed by smooth compactly supported functions with the covariance function

$$\mathbb{E}[\xi(\varphi_1)\xi(\varphi_2)] = \langle \varphi_1, \varphi_2 \rangle_{L^2} .$$

The Hölder regularity of a distribution can be quantified by the following definition.

Definition I.24. Let $\alpha \in \mathbb{R}$. We say that a distribution f is (locally) α-Hölder, or $f \in \mathcal{C}^\alpha_{\mathrm{loc}}(\mathbb{R}^d)$, if $\|f\|_{\mathcal{C}^\alpha(K)} < \infty$ for every compact $K \subset \mathbb{R}^d$, where the norm is defined by:

- for $\alpha < 0$,

$$\|f\|_{\mathcal{C}^\alpha(K)} := \sup_{\eta \in \mathscr{B}^r} \sup_{\lambda \in (0,1]} \sup_{x \in K} \frac{\left|\left\langle f, \eta_x^\lambda \right\rangle\right|}{\lambda^\alpha}; \tag{I.23}$$

- for $\alpha \geq 0$,

$$\|f\|_{\mathcal{C}^\alpha(K)} := \sup_{x \in K} |\langle f, \eta_x \rangle| + \sup_{\eta \in \mathscr{B}^r_{\lfloor \alpha \rfloor}} \sup_{\lambda \in (0,1]} \sup_{x \in K} \frac{\left|\left\langle f, \eta_x^\lambda \right\rangle\right|}{\lambda^\alpha}. \tag{I.24}$$

Here, r is any integer with $r > |\alpha|$ and $\mathscr{B}^r$ denotes the set of test functions $\eta \in C_c^\infty(B(0,1))$ such that $\|\eta\|_{C^r} \leq 1$; $\mathscr{B}^r_k$ is the collection of elements $\eta \in \mathscr{B}^r$ such that $\int_{\mathbb{R}^d} \eta(x)p(x) = 0$ for all polynomials of degree up to k; for all $\eta \in C_c^\infty$, $\eta_x^\lambda(y) = \lambda^{-d}\eta(\frac{y-x}{\lambda})$.

Remark I.25. It is well-known that almost surely ξ is locally a Hölder distribution of regularity strictly less than $-d/2$, i.e., for all $\alpha < -d/2$, $\xi \in \mathcal{C}^\alpha_{\mathrm{loc}}(\mathbb{R}^d)$ a.s.. This fact can be proved via the observation that $\mathbb{E}[\left\langle \xi, \eta_x^\lambda \right\rangle^2] = \lambda^{-d} \|\eta\|_{L^2}^2$ and a wavelet argument similar to the proof of Komogorov continuity theorem.

Let us now consider the continuous Anderson Hamiltonian $\mathcal{H} = -\Delta + \xi$, i.e., take $\mathbf{A} = 0$ and $V = \xi$. In this case, (I.12) then becomes

$$\partial_t u = \Delta u - \xi u \tag{I.25}$$

while (I.11) reads

$$(-\Delta + \xi + a)u = g. \tag{I.26}$$

Due to the fact that ξ is only distribution-valued (with low Hölder regularity), the product ξu in both of the above equations is ill-defined for $d \geq 2$. To see this, we take the elliptic equation I.26 as an example: by taking the mild construction introduced in section I.3.2, one looks for the fixed point u of the equation

$$u = K_a * (g - \xi u), \tag{I.27}$$

where K_a is the Green function for $(-\Delta + a)^{-1}$. Recall that ξ is locally of regularity $\alpha < -d/2$. As a matter of fact, Young's integration theory asserts that the product between two distributions in Hölder space is well-defined if and only if the sum of their Hölder regularities is positive. If the product ξu were to be defined, then its Hölder regularity could not be better than α. The kernel K_a improving the Hölder regularity by 2, it follows that the fixed point of (I.27) (if exists at all) should have regularity at most $2 + \alpha$. Plugging back the regularity and inspecting the product ξu, one observes that ξu is well-defined if and only if $\alpha + (2 + \alpha) > 0$, which is true only when $d = 1$. For $d \geq 2$, the product ξu does not have a meaning at all.

The SPDEs with such ill-posed products are called *singular SPDEs* and constitute a class of equations subject to intensive studies in recent years. In particular, the above parabolic equation (I.25) is one of the archetype of singular SPDE and is given the name of *parabolic Anderson model* (PAM). Let us now give a definition to fix the terminology:

Definition I.26. A random Schrödinger operator $\mathcal{H}$ is said to be *singular*, if one of the associated equations (I.11), (I.12) is a singular SPDE.

The singular SPDE community has seen enormous breakthrough in the past decade and is now equipped with an arsenal of tools. Two of the most successful theories in the field is that of regularity structures [Hai14] due to Martin Hairer and that of para-controlled distributions due to Gubinelli, Imkeller and Perkowski [GIP15]. Both of the above theories incorporate the idea of *renormalization* to deal with the ill-posed products between distributions: it consists in regularizing the noise ξ by convoluting with a mollifier $\rho_\varepsilon = \varepsilon^{-d}\rho(\cdot/\varepsilon)$, then *adding suitably chosen diverging constants* to the equation associated to smoothened noise $\xi_\varepsilon := \xi * \rho_\varepsilon$, and finally sending ε to 0. More precisely, in our case of random operators, it basically consists in constructing the operator

$$\mathcal{H}_\varepsilon = (i\nabla + \mathbf{A})^2 + \xi_\varepsilon + C_\varepsilon$$

where C_ε is a diverging constant, suitably chosen so that the limit $\mathcal{H}_\varepsilon$ exists and is non-trivial.

Before reviewing known results carried out by this renormalization process, let us point out that these theories come with a restriction on the regularity of the noise that depends on the equation at stake (this is the notion of criticality in [Hai14, Assumption 8.3]): for the operator of type (I.1), this prevents from considering white noise in dimension $d \geq 4$.

The aforementioned program of operator construction has been carried out under various conditions:

1. For continuous Anderson case ($\mathbf{A} = 0, V = \xi$) in finite volume (a torus or a bounded box):
   - Fukushima and Nakao [FN77] are the first to consider the one-dimensional continuous Anderson Hamiltonian. The construction introduced in section I.2.3 is due to them.
   - Allez and Chouk [AC15] gave the first construction of continuous Anderson Hamiltonian in the singular regime. It is carried out on a torus (periodic boundary condition) in dimension 2 by paracontrolled distribution.
   - Gubinelli, Ugurcan and Zachhuber [GUZ20] generalized the construction of Allez and Chouk to dimension 3.
   - Chouk and van Zuijlen [Cv21] obtained a construction with Dirichlet boundary condition in dimension 2 in the paracontrolled framework.
   - Labbé [Lab19] provided a unified construction for the continuous Anderson Hamiltonian in all dimensions $d = 1, 2, 3$ with periodic and Dirichlet boundary conditions, using the theory of regularity structures. Let us mention that this construction is based on the mild formulation introduced in section I.3.2 (Construction I).
   - In [Mv22], Matsuda and van Zuijlen provided another unified construction for dimensions $d = 1, 2, 3$ and with periodic and Dirichlet boundary conditions. Their construction is based on the weak formulation introduced in section I.3.2 (Construction II) along with the theory of regularity structures.
2. For magnetic singular random operators ($\mathbf{A} \neq 0$) in finite volume, the only available literature to the best of our knowledge is due to Morin and Mouzard [MM21], where they considered the Hamiltonian of a particle submitted to a white noise magnetic field (i.e. $\operatorname{curl} \mathbf{A} = (0, 0, \xi)$, $V = 0$) in a two-dimensional torus, using the paracontrolled calculus.
3. With Gaussian white noise on full space $\mathbb{R}^d$:
   - Ugurcan [Ugu22] introduced the first construction of continuous Anderson Hamiltonian on full space $\mathbb{R}^2$. His idea is based on a decomposition of noise into singular part and unbounded part, where the former is dealt with using the theory of paracontrolled distributions and the latter can be taken into account by a classical theorem due to Faris and Lavine. However, we believe

there is a gap in the proof which we discovered when trying to adapt this argument to the magnetic case in the writing of Chapter IV. See the discussion in section I.5.

- Ueki [Uek23] proposed a construction of continuous Anderson Hamiltonian on full $\mathbb{R}^2$ using a variant of paracontrolled calculus based on the heat kernels.
- In Chapter III, which is a collaboration with Labbé [HL24], we propose a construction of continuous Anderson Hamiltonian based on the solution theory of PAM in dimensions 2 and 3. This is based on a generalization of the idea introduced in section I.3.2 (Construction III). In dimension 2, we use the trick of exponential transformation introduced in [HL15] which spares us from sophisticated singular SPDE theories, and therefore our construction is much more elementary compared to [Uek23]. In dimension 3, the construction is new and based on a previous work of Hairer and Labbé [HL18] which necessitates the theory of regularity structure.
- In Chapter IV, we try to extend the idea of Ugurcan to the case of uniform magnetic field (i.e., $\operatorname{curl} \mathbf{A} = (0, 0, B)$ for $B > 0$ and $V = \xi$). As mentioned previously, there is still a gap in the argument in applying the Faris-Lavine theorem.

Up to now, we have only talked about the construction of operators. In comparison, available information about the spectral properties of random operators (I.1) driven by white noise is fairly limited in the literature. In a bounded box $(-L/2, L/2)^d$ of side length L, we do know however that the above constructions yield an operator $\mathcal{H}_L$ of compact resolvent, implying that it admits a purely discrete spectrum:

$$\lambda_{1,L} \leq \lambda_{2,L} \leq \lambda_{3,L} \leq \cdots .$$

Since $\mathcal{H}_L$ is a random Schrödinger operator in the sense of Defintion I.18, each $\lambda_{n,L}$ is automatically a random variable. Interests have been paid to understand the almost sure asymptotic of λ_L as $L \to \infty$:

1. For one-dimensional Anderson Hamiltonian defined on bounded interval, McKean [McK94] has proved that almost surely $\lambda_{1,L} \sim a_L = -(\frac{8}{3} \log L)^{2/3}$ as $L \to \infty$. Moreover, he also understood the fluctuation of the rescaled eigenvalue: the random variable $-4\sqrt{a_L}(\lambda_{1,L} + a_L)$ converges in law to the Gumbel's distribution.

2. In a series of work on one-dimensional Anderson Hamiltonian defined on bounded interval, Dumaz and Labbé [DL20, DL23, DL24b] obtained detailed description on the asymptotic behavior of the spectrum in different scaling regimes. They also proved Anderson localization [DL24a].

3. Based on their construction of two-dimensional continuous Anderson Hamiltonian with Dirichlet b.c., Chouk and van Zuijlen [Cv21] extended the asymptotic result of McKean to dimension two: almost surely, $\lambda_{1,L} \sim -C \log L$ with C being the optimal constant in Gagliardo-Nirenberg inequality.

4. In Chapter II, which is a collaboration with Labbé [HL22], we obtain the asymptotic of continuous Anderson Hamiltonian eigenvalues for general dimensions $d = 1, 2, 3$ based on the construction of [Lab19]. We show that $\lambda_{1,L} \sim -(C_d \log L)^{\frac{1}{2-d/2}}$ where C_d is related to the optimal constant of Gagliardo-Nirenberg inequality. This agrees with the result of McKean in dimension 1 and that of Chouk and van Zuijlen in dimension 2. The asymptotic in dimension 3 is new.

5. In Chapter IV, we show that the two-dimensional Landau Hamiltonian with uniform magnetic field and white noise potential defined on bounded box $(-L/2, L/2)^2$ exhibits the same asymptotic as the $2d$ Anderson Hamiltonian obtained by Chouk and van Zuijlen.

Concerning the spectral result of operators on full space $\mathbb{R}^d$, the following is known:

1. Dumaz and Labbé [DL24a] showed that the one-dimensional Anderson Hamiltonian on full space has for spectrum the whole real line. Moreover, Anderson localization holds: that is, the spectrum is pure point and eigenfunctions decay exponentially.

2. Based on his construction on full space $\mathbb{R}^2$, Ueki [Uek23] showed that the spectrum of two-dimensional Anderson Hamiltonian corresponds to the whole real line.

3. In Chapter III, which is a collaboration with Labbé [HL24], we prove that the spectrum of Anderson Hamiltonian on $\mathbb{R}^2$ and $\mathbb{R}^3$ coincides with the whole real line. In dimension 2, our proof follows a different reasoning than that of Ueki and appears to be simpler (see the discussion in section I.5). The result in dimension 3 is new.

Other than the spectrum itself, another quantity of interest is the integrated density of states (IDS). Recall the Anderson Hamiltonian $\mathcal{H}_L$ defined on a bounded box of size length L has purely discrete spectrum $\lambda_{1,L} \leq \lambda_{2,L} \leq \cdots$. One can therefore define the quantity

$$N_L(\lambda) = \frac{1}{L^d} \sum_{n=1}^{\infty} \mathbf{1}_{\lambda_{n,L} \leq \lambda}.$$

With the use of an ergodic theorem, for each fixed $\lambda \in \mathbb{R}$, it is possible to show that $N_L(\lambda)$ converges a.s. to a non-random quantity $N(\lambda)$ as $L \to \infty$. The function $\lambda \mapsto N(\lambda)$ is called the integrated density of states: this is because its derivative $n(\lambda) = \frac{\mathrm{d}N(\lambda)}{\mathrm{d}\lambda}$, if it exists, reflects how dense the energy states are around the energy level λ, per unit volume. Consequently, the IDS provides information on the spectrum of the operator defined on the full space. However, the mere construction of IDS in the case of white noise is non-trivial.

1. Matsuda [Mat21] constructed the IDS for two-dimensional Anderson Hamiltonian by relating the Laplace transform of IDS to the solution of PAM. This allows him to prove the Lifshitz tail $\log N(\lambda) \sim \lambda/C$, $\lambda \to -\infty$, by using a Feyman-Kac

argument to relate the solution PAM to the self-intersection local time of Brownian bridge.

2. Matsuda and van Zuijlen extended the construction of IDS for general dimensions $d = 1, 2, 3$. They also generalized the previous Lifshitz tail based the asymptotic that we obtained in [HL22].

I.4 NOTIONS OF REGULARITY STRUCTURES

In the subsequent chapters, the knowledge of regularity structures plays an essential role. For this reason, the current section is devoted to the introduction of this theory. The goal is to illustrate how the concept of regularity structures allows to solve SPDEs of type (I.25) or (I.26) and to construct singular operators.

In this section, we should focus on the case without magnetic field, $\mathbf{A} = 0$. To fix the idea, we consider the equation (I.26) on the torus $\mathbb{T}^d$ for a fixed $g \in L^2(\mathbb{T}^d)$. More precisely, we aim to show that there exists a constant $a > 0$ large enough such that the solution to

$$-\Delta u + \xi u + au = g, \quad \text{on } \mathbb{T}^d \tag{I.28}$$

exists.

As mentioned in section I.3.4, due to the ill-defined product ξu, the equation should be understood in the sense of renormalization: That is, we want to find a suitable diverging constant C_ε such that the solution u_ε to

$$-\Delta u_\varepsilon + \xi_\varepsilon u_\varepsilon + C_\varepsilon u_\varepsilon + au_\varepsilon = g, \quad \text{on } \mathbb{T}^d \tag{I.29}$$

converges to a meaningful limit u as $\varepsilon \to 0$. Here, $\xi_\varepsilon = \xi * \rho_\varepsilon$ is the white noise regularized by the mollifier $\rho_\varepsilon = \varepsilon^{-d}\rho(\frac{\cdot}{\varepsilon})$, where ρ is some fixed smooth and compactly supported function. We write (I.29) into the mild formulation

$$u_\varepsilon = K_a * [g - (\xi_\varepsilon + C_\varepsilon)u_\varepsilon]. \tag{I.30}$$

The idea of regularity structures is to *lift* (I.30) to an abstract setting where one can define products without problem:

$$\mathrm{u} = \mathcal{K}\mathrm{g} - \mathcal{K}(\mathrm{u}\Xi) \tag{I.31}$$

where we have the following correspondence:

ξ_ε	$\rightsquigarrow$	Ξ	Abstract symbol for the noise.
u_ε, g	$\rightsquigarrow$	u, g	Abstract functions taking values in linear combinations of symbols.
K_a	$\rightsquigarrow$	$\mathcal{K}$	Integral kernel acting on abstract functions.

In short, to concretize this idea into a consistent theory, one first needs a sufficiently rich collection of abstract symbols, such as Ξ for the noise and X^k for the polynomials, which generate a vector space $\mathcal{T}$ (*model space*, Definition I.27). These symbols will describe the local expansion of a given real-valued function via a *model* (Π, Γ) (Defintion I.28), in a way to be elaborated in section I.4.1 below. These local descriptions of functions give rise to a notion of abstract function space $\mathscr{D}$ containing elements like u and g, which are functions from $\mathbb{R}^d$ into $\mathcal{T}$ (*modelled distributions*, Definition I.29). Moreover, we need a *realization operator* $\mathcal{R}$ (Theorem I.33) which allows to recover from these abstract functions in $\mathscr{D}$ to a real-valued function/distribution in a Hölder or Sobolev space. This realization operator $\mathcal{R}$ should be consistent with the abstract integral kernel $\mathcal{K}$ in the following sense:

$$K_a * \mathcal{R}\mathrm{u} = \mathcal{R}\mathcal{K}\mathrm{u}.$$

Provided that all the above be true, the resolution of (I.28), as well as the construction of the operator $\mathcal{H} = (i\nabla + \mathbf{A})^2 + \xi$, can then be represented by the diagram in Figure I.1.

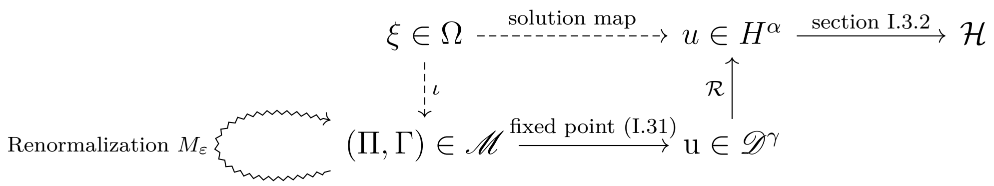

Figure I.1: Construction of random operator by regularity structure.

In Figure I.1, the space Ω denotes the canonical probability space on which lives the realizations of ξ, while $\mathscr{M}$ denotes the collection of models which are *admissible* (Definition I.36). Notice here that the renormalization is represented by a map M_ε acting on the set $\mathscr{M}$ of admissible models: upon reconstruction, this will eventually give rise to the diverging constant C_ε in the equation (I.29) (see section I.4.4).

Let us point out that in the diagram the maps represented by arrows $\rightarrow$ are all continuous, whereas the dashed ones $\dashrightarrow$ are only measurable. As a consequence, the above diagram implies the operator $\mathcal{H}$ is a measurable function of ξ, thus a random operator in the sense of Defintion I.18.

In the sequel, the objective is to motivate and make rigorous Figure I.1: The definitions of regularity structure, models, modelled distributions, and the reconstruction theorem are motivated and given in section I.4.1. section I.4.2 illustrates the general reasoning to obtain a regularity structure based on a given SPDE problem. In section I.4.3, we introduce the admissible models, which are the models designed to be compatible with the kernel K_a, and then comment on the existence of fixed point for (I.31). Finally, the renormalization procedure is discussed in section I.4.4.

I.4.1 A generalized theory of Taylor expansions

The starting point of the abstract theory comes from the Taylor expansions. Suppose the function $u : \mathbb{R} \to \mathbb{R}$ is γ-Hölder function with $\gamma \in (n, n+1)$, $n \in \mathbb{N}$. That is, the function u is n-times differentiable and the n^{th}-derivative $u^{(n)}$ verifies the bound $|u^{(n)}(x) - u^{(n)}(y)| \leq C|x-y|^{\gamma-n}$ uniformly for $x, y \in \mathbb{R}$. The Taylor-Young theorem provides a way to approximate u locally by a power series centered at $x_0 \in \mathbb{R}$ with an error of order $O(|x-x_0|^\gamma)$: For $x > x_0$, one has

$$u(x) = a_0(x_0) + a_1(x_0)(x-x_0) + \cdots + a_\gamma(x_0)(x-x_0)^n + R_\gamma(x), \tag{I.32}$$

where $a_k(x_0) = u^{(k)}(x_0)/(k!)$ and

$$R_\gamma(x) = \int_{x_0}^x \frac{(x-t)^{n-1}}{(n-1)!}(f^{(n)}(t) - f^{(n)}(x_0))\,\mathrm{d}t.$$

Indeed, since $f^{(n)}$ is $(\gamma - n)$-Hölder, the remainder term $R_\gamma(x)$ satisfies the bound $|R_\gamma(x)| \leq C|x-x_0|^\gamma$.

We can put (I.32) in an abstract setting by considering the abstract monomials $\mathbf{1}, X, X^2, X^3, \ldots$ (with the convention $X^0 = \mathbf{1}$) in the following way. Define the maps:

$$\Pi_{x_0}(X^k)(x) = (x-x_0)^k, \quad \Gamma_{x_0,y_0}X^k = (X + x_0 - y_0)^k, \quad \forall x, x_0, y_0 \in \mathbb{R}, k \in \mathbb{N}$$

and extend linearly to all abstract polynomials $p(X) = a_0 + a_1 X + \cdots + a_k X^k$. Then a natural way to *lift* u is to define

$$\mathrm{u}(x_0) = a_0(x_0)\mathbf{1} + a_1(x_0)X + a_2(x_0)X^2 + \cdots + a_n(x_0)X^n$$

and u is related to u by the identity

$$u(x) = \Pi_{x_0}[\mathrm{u}(x_0)](x) + R_\gamma(x).$$

In other words, $\mathrm{u}(x_0)$ encodes the *coefficients* of the Taylor expansion of u at reference point x_0 up to order $n = \lfloor \gamma \rfloor$ while Π_{x_0} sends an abstract polynomial to a real polynomial centered at x_0. On the other hand, the map Γ_{x_0,y_0} transforms an abstract expansion centered at y_0 to that centered at x_0: this can be seen from the identity

$$\Pi_{y_0}\mathrm{u}(y_0) = \Pi_{x_0}\Gamma_{x_0,y_0}\mathrm{u}(y_0), \quad \forall x_0, y_0 \in \mathbb{R}.$$

Moreover, the abstract function u also encodes the fact that u is γ-Hölder: one can check that u is γ-Hölder if and only if

$$\left| u^{(m)}(x+h) - \sum_{k=m}^{n} \frac{u^{(m+n)}(x)}{k!}h^k \right| \lesssim |h|^{\gamma-m}, \quad m = 0, \ldots, n,$$

which we can show is equivalent to

$$\|\mathrm{u}(x+h) - \Gamma_{x+h,x}\mathrm{u}(x)\|_m \lesssim |h|^{\gamma-m}, \quad m = 0, \ldots, n,$$

where $\|\tau\|_m$ denotes the absolute value of the coefficient for the term X^m in each abstract polynomial τ.

This inspires the following definitions

Definition I.27. A *regularity structure* is a triplet $(A, \mathcal{T}, G)$ satisfying:

1. The index set A is a locally finite subset of $\mathbb{R}$ containing 0 and is bounded from below.
2. The model space $\mathcal{T}$ is a $\mathbb{R}$-vector space with the graded structure
$$\mathcal{T} = \bigoplus_{\alpha \in A} T_\alpha,$$
with each $\mathcal{T}_\alpha$ being a Banach space and $\mathcal{T}_0$ isomorphic to $\mathbb{R}$.
3. The structure group G is a collection of linear maps Γ on $\mathcal{T}$ such that for all $\alpha \in A$ and $\tau \in \mathcal{T}_\alpha$,
$$\Gamma\tau - \tau \in \bigoplus_{\beta < \alpha} T_\alpha,$$
and that $\Gamma\mathbf{1} = \mathbf{1}$, where $\mathbf{1}$ is the unit vector of $\mathcal{T}_0$.

Definition I.28. We say that the couple $Z = (\Pi, \Gamma)$ is a *model* for the regularity structure $(A, \mathcal{T}, G)$ if

- $\Gamma : \mathbb{R}^d \times \mathbb{R}^d \to G$ maps $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ to an element $\Gamma_{x,y} \in G$ (corresponding to the change of reference point from y to x.)
- $\Pi = (\Pi_x)_{x \in \mathbb{R}^d}$ is a family of maps where $\Pi_x : \mathcal{T} \to \mathcal{S}'(\mathbb{R}^d)$ (which sends any symbol $\tau \in \mathcal{T}$ to its real Taylor expansion at point x.)
- (Structure equations) $\Gamma_{x,y}\Gamma_{y,z} = \Gamma_{x,z}$ and $\Pi_y = \Pi_x \circ \Gamma_{xy}$.
- (Analytical bounds) Fix a compact $K \subset \mathbb{R}^d$. It holds
$$|\left\langle \Pi_x \tau, \varphi_x^\lambda \right\rangle| \lesssim \|\tau\|_\alpha \lambda^\alpha, \quad \|\Gamma_{x,y}\tau\|_\beta \lesssim \|\tau\|_\alpha |x - y|^{\alpha-\beta}$$
uniformly for all $x, y \in K$, all $\lambda \in (0, 1]$, all $\alpha \in A$, all $\tau \in \mathcal{T}_\alpha$, all $\beta < \alpha$ and all test functions φ supported in $B(0, 1)$ with $\|\varphi\|_{C^r} \leq 1$. (Here, r is any integer such that $r > |\alpha|$.)

In the preceding discussion we have already seen a first example of regularity structure and model: Let $(A, \bar{\mathcal{T}}, G)$ be such that

$$\begin{aligned}
&\bar{A} = \mathbb{N},\\
&\bar{\mathcal{T}} = \bigoplus_{k \in \bar{A}} \bar{\mathcal{T}}_k, \quad \text{where } \bar{\mathcal{T}}_k = \operatorname{span}\left\{X^k\right\},\\
&\bar{G} = \{\Gamma_h : h \in \mathbb{R}\}, \quad \text{where } \Gamma_h X^k = (X + h)^k \text{ and extends linearly to all } \mathcal{T}.
\end{aligned}$$

One can check that G is indeed a group with the group multiplication being the composition of maps. The model (Π, Γ) is defined exactly as above, with $\Gamma_{x,y} := \Gamma_{x-y} \in G$.

We now provide an analogous notion of Hölder functions for maps taking values in the model space $\mathcal{T}$.

Definition I.29 (Modelled distributions)**.** Let $\gamma \in \mathbb{R}$. Given a model (Π, Γ) on a regularity structure $(A, \mathcal{T}, G)$, define the family $\mathscr{D}^\gamma = \mathscr{D}^\gamma(\Pi, \Gamma)$ of all maps $\mathrm{f} : \mathbb{R}^d \to \oplus_{\alpha<\gamma}\mathcal{T}_\alpha$ such that for all compact set $K \subset \mathbb{R}^d$

$$\|\!|\mathrm{f}|\!\|_{\gamma;K} := \sup_{x\in K}\sup_{\beta<\gamma} \|\mathrm{f}(x)\|_\beta + \sup_{|x-y|\leq 1}\sup_{\beta<\gamma} \frac{\|\mathrm{f}(x) - \Gamma_{x,y}\mathrm{f}(y)\|_\beta}{|x-y|^{\gamma-\beta}} < \infty \tag{I.33}$$

When comparing two modelled distributions $\mathrm{f}, \bar{\mathrm{f}}$ associated to two different models (Π, Γ), $(\bar{\Pi}, \bar{\Gamma})$, we use the following quantity

$$\left\|\!\left| \mathrm{f}; \bar{\mathrm{f}} \right|\!\right\|_{\gamma;K} := \sup_{x\in K}\sup_{\beta<\gamma} \left\|(\mathrm{f}-\bar{\mathrm{f}})(x)\right\|_\beta + \sup_{|x-y|\leq 1}\sup_{\beta<\gamma} \frac{\left\|\mathrm{f}(x) - \bar{\mathrm{f}}(x) - \Gamma_{x,y}\mathrm{f}(y) + \bar{\Gamma}_{x,y}\bar{\mathrm{f}}(y)\right\|_\beta}{|x-y|^{\gamma-\beta}}.$$

Remark I.30*.* Note that the notion of modelled distributions depends on the model given. For this reason, the quantity $\left\|\!\left| \mathrm{f}; \bar{\mathrm{f}} \right|\!\right\|_{\gamma;K}$ is not a function of $\mathrm{f} - \bar{\mathrm{f}}$ but also depends on their respective models.

Remark I.31*.* When $(A, \mathcal{T}, G)$ is such that $\min(A\backslash\mathbb{N}) = \alpha \in \mathbb{R}$, we sometimes denote by $\mathscr{D}^\gamma_\alpha$ the modelled distributions taking values in $\mathcal{T}$.

We have seen in the above discussion that the following is true on the polynomial regularity structure $(\bar{A}, \bar{\mathcal{T}}, \bar{G})$:

Proposition I.32. *Let* $\gamma > 0$ *and* $\mathscr{D}^\gamma$ *be the space of modelled distribution on* $(\bar{A}, \bar{\mathcal{T}}, \bar{G})$ *with respect to the model* (Π, Γ)*. A function* $u : \mathbb{R} \to \mathbb{R}$ *is locally* γ*-Hölder if and only if there exists* $\mathrm{u} : \mathbb{R} \to \bar{\mathcal{T}}$ *such that* $\mathrm{u} \in \mathscr{D}^\gamma$ *and* $\langle \mathbf{1}^*, \mathrm{u}(x)\rangle = u(x)$ *for all* $x \in \mathbb{R}$*. Here* $\langle \mathbf{1}^*, \cdot\rangle$ *denotes the dual of* $\mathbf{1}$*, i.e.,* $\langle \mathbf{1}^*, \cdot\rangle : \bar{\mathcal{T}} \to \mathbb{R}$ *is such that* $\left\langle \mathbf{1}^*, X^k \right\rangle = \delta_{k,0}$ *with* $\delta_{k,0}$ *being the Kronecker delta.*

The following theorem assures that one can always associate to every modelled distribution in $\mathscr{D}^\gamma$ a unique, *bona fide* Hölder distribution on $\mathbb{R}^d$.

Theorem I.33 (Reconstruction)**.** *Fix a regularity structure* $(A, \mathcal{T}, G)$ *and a model* (Π, Γ)*. Let* $\alpha \leq 0 < \gamma$ *such that* $\alpha \notin \mathbb{N}$*. Then, there exists a unique map* $\mathcal{R} : \mathscr{D}^\gamma_\alpha \to \mathcal{C}^\alpha_{\mathrm{loc}}$ *such that for all* $\mathrm{f} \in \mathscr{D}^\gamma_\alpha$

$$\left|(\mathcal{R}\mathrm{f} - \Pi_x\mathrm{f}(x))(\eta^\lambda_x)\right| \lesssim \|\Pi\|_\gamma \|\!|\mathrm{f}|\!\|_{\gamma;K}\lambda^\gamma \tag{I.34}$$

uniformly for all $\eta \in \mathcal{B}^r_0$*,* $\lambda \in (0, 1]$ *and* x *in a compact* $K \subset \mathbb{R}^d$*. The map* $\mathcal{R}$ *is called the* reconstruction operator*.*

Moreover, if the expression of f *does not contain terms of homogeneity less than* 0*, one has* $\mathcal{R}\mathrm{f}(x) = \langle \mathbf{1}^*, \mathrm{f}(x)\rangle = (\Pi_x\mathrm{f}(x))(x)$ *for all* $x \in \mathbb{R}^d$*.*

Remark I.34*.* We should stress that the operator $\mathcal{R}$ depends on the underlying models (Π, Γ). Moreover, this dependency is continuous in a suitable sense.

Remark I.35. In fact, in order to make rigorous Figure I.1, one needs a twist to the theory introduced above. Indeed, the modelled distribution in Definition I.29 and the corresponding Reconstruction Theorem I.33 is in the flavor of Hölder space $\mathcal{C}^\alpha$, instead of the Sobolev-type space H^α that we need for the construction of operators. To remedy this inconsistency, one should really replace the norm (I.33) by

$$\|\mathrm{f}\|_{\mathscr{D}^\gamma_{p,q}} := \sup_{\beta<\gamma} \left[\int_{\mathbb{R}^d} \|\mathrm{f}(x)\|_\beta^p \,\mathrm{d}x\right]^{\frac{1}{p}} + \sup_{\beta<\gamma} \left\{\int_{B(0,\sqrt{a})} \left[\int_{\mathbb{R}^2} \left(\frac{\|\mathrm{f}(x+h) - \Gamma_{x+h,x}\mathrm{f}(x)\|_\beta}{|h|^{\gamma-\beta}}\right)^p \mathrm{d}x\right]^{\frac{q}{p}} \frac{\mathrm{d}h}{h^d}\right\}^{1/q}$$

which is in the flavor of the Besov space $\mathcal{B}^\gamma_{p,q}$. For the space of modelled distributions so-defined, there exists a unique, continuous reconstruction operator $\mathcal{R} : \mathscr{D}^\gamma_{p,q} \to \mathcal{B}^\alpha_{p,q}$ with $\alpha = \min(A\backslash\mathbb{N})$. See [Lab19] and also [HL17] for details.
Notice also that the integral over h is taken over a ball with radius $\sqrt{a}$ instead of the unit ball: the choice is due to the fact that we tailored the space with respect to the integral kernel K_a which decays as $\exp(-\sqrt{a}|x|)$ for $|x|$ large (whence the singularity at 0 is dominant only when $|x| \leq \frac{1}{\sqrt{a}}$). This setting allows us to have the embedding

$$\|f\|_{\mathscr{D}^{\gamma-\kappa}_{p,q}} \lesssim a^{-\kappa/2} \|f\|_{\mathscr{D}^\gamma_{p,q}}$$

for all $\kappa > 0$, which will be useful when we build the fixed point in section I.4.3.

I.4.2 Construction of a regularity structure

We now have our first regularity structure $(\bar{A}, \bar{\mathcal{T}}, \bar{G})$, but this is not sufficient in order to formulate an abstract version of the problem (I.30). Indeed, we want to enrich the model space $\bar{\mathcal{T}}$ by additional symbols. In particular, the desired model space $\mathcal{T}$ should at least include the symbol Ξ, representing the noise, as well as $\mathcal{I}(\Xi)$, representing what one would obtain by applying the convolution kernel on the noise. Finally, $\mathcal{T}$ should admits a corresponding structure group G and index set A such that $(A, \mathcal{T}, G)$ satisfies the condition in Definition I.27. The existence of a corresponding structure group G is not at all trivial.

To achieve this generalization, one needs a better understanding of the algebraic structures at stake. It turns out that, provided the SPDE problem satisfies the *subcritical assumption* [Hai14, Assumption 8.3] (which is the case for (I.28) in dimensions $d \leq 3$), one can always build a corresponding regularity structure $(A, \mathcal{T}, G)$. This is the content of [Hai14, Section 8.1]. Here, although it is not needed for the subsequent chapters, we will give an elementary introduction to the construction of the structure group G.

We first take a closer look at the structure group $\bar{G}$ of polynomials. Note that the polynomial model space $\bar{\mathcal{T}}$ comes with a structure of *Hopf algebra*. One writes $(\bar{\mathcal{T}}, \mathcal{M}, \mathbf{1}, \Delta, \mathbf{1}^*, \mathcal{A})$, where

- $\mathcal{M} : \bar{\mathcal{T}} \otimes \bar{\mathcal{T}} \to \bar{\mathcal{T}}$ is the multiplication: $\mathcal{M}(X^k \otimes X^\ell) = X^{k+\ell}$.
- $\mathbf{1} = X^0$ is the unit element in $\bar{\mathcal{T}}$.
- $\Delta : \bar{\mathcal{T}} \to \bar{\mathcal{T}} \otimes \bar{\mathcal{T}}$ is the comultipication: for all $n \in \mathbb{N}$,

$$\Delta X^n = \sum_{k=0}^{n} \binom{n}{k} X^{n-k} \otimes X^k.$$

- $\mathbf{1}^*$ is the counit: That is $\left\langle \mathbf{1}^*, X^k \right\rangle = \delta_{k,0}$ with $\delta_{k,0}$ being the Kronecker delta.
- $\mathcal{A} : \bar{\mathcal{T}} \to \bar{\mathcal{T}}$ is the antipode: $\mathcal{A}X^k = (-X)^k$.

One can check that these maps verify the following properties

1. $(\bar{\mathcal{T}}, \mathcal{M}, \mathbf{1})$ forms an associative algebra.
2. $(\bar{\mathcal{T}}, \Delta, \mathbf{1}^*)$ forms a coassociative coalgebra.
3. $\Delta, \mathbf{1}^*$ are algebra morphisms for $(\bar{\mathcal{T}}, \mathcal{M}, \mathbf{1})$ and $\mathcal{M}, \mathbf{1}$ are coalgebra morphisms for $(\bar{\mathcal{T}}, \Delta, \mathbf{1}^*)$.
4. $\mathcal{A}$ verifies the property $\mathcal{M}(\mathcal{A} \otimes I)\Delta = \langle \mathbf{1}^*, \cdot \rangle \, \mathbf{1} = \mathcal{M}(I \otimes \mathcal{A})\Delta$.

We say that $\bar{\mathcal{T}}$ is a Hopf algebra given that the above four properties hold. (See [Kas95, Chapter III] for a general treatment of Hopf algebra.)

To understand the behavior of coefficients in a change of base point, one notices that

$$\Gamma_h X^k = (X + h)^k = (f_h \otimes I)\Delta X^k, \tag{I.35}$$

where $f_h : \bar{\mathcal{T}} \to \mathbb{R}$ is defined by $f_h(X^\ell) = h^\ell$ for all $\ell \in \bar{A}$, and we identify $c \otimes \tau$ with $c\tau$ for $c \in \mathbb{R}$. This inspires the interpretation that Δ is a decomposition into the *coefficient part* and *symbol part* in a change of reference point. More precisely, for $\tau \in \bar{\mathcal{T}}_k$, one has

$$\Delta \tau = \sum_{\ell=0}^{k} \underbrace{c_\ell \tau_\ell^{(1)}}_{\text{coefficient}} \otimes \underbrace{\tau_\ell^{(2)}}_{\text{symbol}} .$$

This is due to the following observation: $f_h(\tau_\ell^{(1)})$ encodes the coefficient in front of the term $\tau_\ell^{(2)}$ when re-expanding τ at a reference point translated by h. Notice (I.35) shows that the knowledge of the maps f_h (which "evaluate the coefficient parts at h") characterizes the Group $\bar{G} = (\Gamma_h)_{h\in\mathbb{R}}$. Moreover, the antipode $\mathcal{A}$ provides a way to find the inverse element in the group $\bar{G}$: Indeed, one can check that $f_{-h}(X^k) = (-h)^k = f_h(\mathcal{A}X^k)$ and thus $\Gamma_h^{-1} = \Gamma_{-h} = (f_{-h} \otimes I)\Delta = (f_h \mathcal{A} \otimes I)\Delta$.

More generally, suppose now that we have a graded vector space $\mathcal{T}$ with an Hopf algebra structure $(\mathcal{T}, \mathcal{M}, \mathbf{1}, \Delta, \mathbf{1}^*, \mathcal{A})$, let us illustrate how to construct a structure group

G making $\mathcal{T}$ a regularity structure. We impose the additional conditions that for $\tau_1, \tau_2 \in \mathcal{T}$

$$\Delta\mathcal{M}(\tau_1 \otimes \tau_2) = (\Delta\tau_1) \cdot (\Delta\tau_2), \tag{I.36}$$

where the product $\cdot$ is defined component-wise, and

$$\mathcal{A}\mathcal{M}(\tau_1 \otimes \tau_2) = \mathcal{M}(\mathcal{A}\tau_1 \otimes \mathcal{A}\tau_2). \tag{I.37}$$

For $f : \mathcal{T} \to \mathbb{R}$, we say that f is group-like if

$$f(\mathcal{M}(\tau_1 \otimes \tau_2)) = f(\tau_1) \cdot f(\tau_2), \quad \tau_1, \tau_2 \in \mathcal{T}. \tag{I.38}$$

Given a group-like f, define for all $\tau \in \mathcal{T}$

$$\Gamma_f \tau := (f \otimes I)\Delta\tau. \tag{I.39}$$

We claim that $G = \{\Gamma_f : f \text{ is group-like}\}$ defines a group. Indeed, it holds that $\Gamma_f\Gamma_g = \Gamma_{f\circ g}$ where

$$f \circ g := (g \otimes f)\Delta$$

is indeed group-like by (I.36). Moreover, one has that

$$\begin{aligned}(f\mathcal{A} \otimes I)\Delta\Gamma_f\tau &= (f \otimes f \otimes I)(I \otimes \mathcal{A} \otimes I)(I \otimes \Delta)\Delta\tau \\ &= (f \otimes I)\left(\mathcal{M}(I \otimes \mathcal{A})\Delta \otimes I\right)\Delta\tau \\ &= (f \otimes I)(\mathbf{1}^*(\cdot)\mathbf{1} \otimes I)\Delta\tau = \tau\end{aligned}$$

where in the second equality we used the associative property $(I \otimes \Delta)\Delta = (\Delta \otimes I)\Delta$ and the group-like property (I.38). Similarly, one also has $\Gamma_f(f\mathcal{A} \otimes I)\Delta\tau = \tau$. This shows that

$$\Gamma_f^{-1} = (f\mathcal{A} \otimes I)\Delta.$$

Since $f \circ \mathcal{A}$ is group-like by (I.37), we have proven that G forms a group. One can check that G indeed satisfies Definition I.27.

The above discussion leads us to the following reasoning: We will define a model space $\mathcal{T}$ by collecting necessary symbols for setting up the desired fixed point. If this $\mathcal{T}$ can be identified as an Hopf algebra satisfying (I.36) and (I.37) and if we can identify a family of group-like maps (which determine the "evaluation of coefficients"), then we can construct the structure group G through (I.39). Unfortunately, it is generally not true that $\mathcal{T}$ forms an Hopf algebra; however, it can always be identified as a *comodule* over an Hopf algebra $\mathcal{T}_+$ which *contains all necessary symbols to describe the coefficient parts*, and the previous reasoning can be carried over on $\mathcal{T}_+$. See [Hai14, Section 8.1] for details.

In the sequel, we will consider the regularity structure designed for (I.28). Let $\mathcal{T}$ be given by

$$\begin{aligned}\mathcal{F}_0 &:= \{\Xi\}, \quad \mathcal{U}_0 := \{X^k : k \in \mathbb{N}^2\} \\ \mathcal{F}_{n+1} &:= \{\tau_1\tau_2 : \tau_1 \in \mathcal{F}_n, \tau_2 \in \mathcal{U}_n\}, \quad n \geq 0 \\ \mathcal{U}_{n+1} &:= \{\mathcal{I}(\tau) : \tau \in \mathcal{F}_n\} \cup \mathcal{U}_0, \quad n \geq 0 \\ \mathcal{T} &:= \operatorname{span}\left\{\bigcup_{n\geq 0}(\mathcal{F}_n \cup \mathcal{U}_n)\right\}.\end{aligned}$$

Here, the symbol $\mathcal{I}(\tau)$ should be understood as *the abstract convolution kernel $\mathcal{K}$ applied to τ, but with its Taylor series removed.* For this reason, $\mathcal{I}(X^k)$ always vanishes. This map $\mathcal{I}$ is called the *abstract integration map.* See (I.42) below for its precise relation with respect to the kernel K_a. Each of the symbol τ obtained in the recursive manner above is assigned a *homogeneity* $|\tau|$: Define $|\mathbf{1}| = 0$, $|X_i| = 1$, $|\Xi| = -\frac{d}{2} - \kappa$; for all $\tau \in \mathcal{T}$, define $|\tau|$ inductively by

$$|\tau_1\tau_2| = |\tau_1| \cdot |\tau_2|, \quad |\mathcal{I}(\tau)| = |\tau| + 2, \quad \forall \tau_1, \tau_2, \tau \in \bigcup_{n\geq 0} (\mathcal{F}_n \cup \mathcal{U}_n).$$

We also define $\mathcal{H}$ to be the smallest algebra containing $\mathcal{T}$ and $\mathcal{H}_+$ the algebra containing symbols of the form[4]

$$X^k \prod_i \mathcal{I}(\tau_i),$$

where $\tau_i \in \mathcal{T}$ such that $|\tau_i| + 2 > 0$. The homogeneity defined above results in an index set A (bounded from below since Ξ is the symbol with the lowest homogeneity, see the subcritical assumption [Hai14, Assumption 8.3]) along with the graded structure $\mathcal{H} = \oplus_{\alpha \in A} \mathcal{H}_\alpha$, where $\mathcal{H}_\alpha$ is the vector space spanned by symbols with homogeneity equal to α. By [Hai14, Section 8.2] (using a reasoning similar to what we described previously), there exists a structure group G making $(A, \mathcal{H}, G)$ a regularity structure. Finally, we can restrict the group action of G on $\mathcal{T}$ and show that $(A, \mathcal{T}, G)$ still satisfies Definition I.27.

In practice, we usually truncate the above $\mathcal{T}$ to have a finite-dimensional model space. For (I.28) as an example, it turns out sufficient to consider symbols with homogeneity lower than $\gamma = \frac{d}{2} + 2\kappa$. The model space is given by

$$\mathcal{T}_{\text{Anderson}} = \begin{cases} \text{span}\{\Xi, \Xi\mathcal{I}(\Xi), X_i\Xi, \mathbf{1}, \mathcal{I}(\Xi), X_i\}, & d = 2, \\ \\ \text{span}\,\{\Xi, \Xi\mathcal{I}(\Xi), \Xi\mathcal{I}(\Xi\mathcal{I}(\Xi)), X_i\Xi, \Xi\mathcal{I}(\Xi\mathcal{I}(\Xi\mathcal{I}(\Xi))), \Xi\mathcal{I}(\Xi X_i) \\ \qquad \mathbf{1}, \mathcal{I}(\Xi), \mathcal{I}(\Xi\mathcal{I}(\Xi)), X_i, \mathcal{I}(\Xi\mathcal{I}(\Xi\mathcal{I}(\Xi))), \mathcal{I}(\Xi X_i)\}\,, & d = 3, \end{cases}$$

and the structure group G is that restricted to $\mathcal{T}_{\text{Anderson}}$.

I.4.3 Admissible models and the abstract convolution kernel

Up to now, we have given the definition of models and discussed the consequences of this definition. However, we have not yet explained how actually can one construct a model tailored to the SPDE problem at stake. In our case, it means that we want to define the couple (Π, Γ) with respect to the regularity structure $(A, \mathcal{T}_{\text{Anderson}}, G)$ given in the previous section.

It turns out that we only need to specify the maps $\Pi = (\Pi_x)_{x\in\mathbb{R}^d}$; the knowledge of Π determines that of Γ via the rigid structure in Definition I.28 in our case. There is a

[4]In [Hai14], the symbol $\mathcal{I}(\tau)$ in $\mathcal{H}_+$ is replaced by the notation $\mathcal{J}(\tau)$ to distinguish the role of $\mathcal{H}_+$ from $\mathcal{H}$.

natural way to construct Π which we describe in the following. The idea is to first define the model Π^ε associated to the regularized noise ξ^ε before passing to limit $\varepsilon \to 0$: let

$$\Pi_x^\varepsilon X^k(y) = (y-x)^k, \quad \Pi_x^\varepsilon \Xi(y) = \xi_\varepsilon(y) \tag{I.40}$$

and by induction, we define

$$\Pi_x^\varepsilon(\tau_1\tau_2) = (\Pi_x^\varepsilon \tau_1)(\Pi_x^\varepsilon \tau_2) \tag{I.41}$$

for all $\tau_1, \tau_2 \in \mathcal{T}_{\text{Anderson}}$. We have not yet specified for symbols of the form $\mathcal{I}(\tau)$: this has to be done in a way "compatible" to the convolution kernel at hand, the Green function K_a for $-\Delta + a$. By translational invariance of $-\Delta$, one can write $K_a(x,y) = K_a(x-y)$ and K_a can be seen as a function defined on $\mathbb{R}^d$. Recall that K_a is everywhere smooth except at the origin, and decays at rate $e^{-\sqrt{a}|x|}$ when $|x|$ goes to infinity. It turns out that the regularizing property of K_a is determined by its behavior close to the singularity, which leads us to write

$$K_a = K_+ + K_-$$

where K_+ is supported in the ball with radius $\frac{1}{\sqrt{a}}$ and K_- is smooth and exponentially decaying. Since all distributions convolved with K_- will result in a smooth function and therefore can already be described by the polynomial regularity structure, we are only interested in the singular part K_+ when defining $\Pi_x^\varepsilon \mathcal{I}(\tau)$. For all $\tau \in \mathcal{T}_{\text{Anderson}}$, set

$$\Pi_x^\varepsilon \mathcal{I}(\tau)(y) = K_+ * \Pi_x \tau(y) - \sum_{k\in\mathbb{N}^2:|k|<|\tau|+2} \frac{(y-x)^k}{k!} \int_{\mathbb{R}^d} D^k K_+(x-z) * \Pi_x^\varepsilon \tau(z)\,\mathrm{d}z. \tag{I.42}$$

Let us comment on the motivation for (I.42). Indeed, what we expect in the end is to build an abstract kernel $\mathcal{K}_+$ in order to make sense of (I.31). In particular, it has to verify the property

$$\mathcal{R}\mathcal{K}_+ \mathrm{f} = K_+ * \mathcal{R}\mathrm{f} \tag{I.43}$$

for all modelled distribution f and the reconstruction operator $\mathcal{R}$ associated to the model $(\Pi^\varepsilon, \Gamma^\varepsilon)$. Suppose for now f consists of a single term τ such that $|\tau|+2>0$, i.e., $\mathrm{f}(x) = \tau$ for all $x \in \mathbb{R}^d$. Fix a base point x and let us look at the right-hand side of the above identity: since $\mathcal{R}\mathrm{f}$ is locally well approximated by the function $w = \Pi_x \mathrm{f}(x)$ (recall the reconstruction bound (I.34)), we write

$$K_+ * \mathcal{R}\mathrm{f} = K_+ * (\mathcal{R}\mathrm{f} - w) + K_+ * w.$$

Moreover, the function $K_+ * w$ is approximated by its Taylor expansion in a neighborhood of x up to an order strictly less than $|\tau|+2$ with a remainder term $R_{|\tau|+2}$:

$$K_+ * w(y) = \sum_{|k|<|\tau|+2} D^k(K_+ * w)(x)\frac{(y-x)^k}{k!} + R_{|\tau|+2}(y).$$

If one looks closely, then one notices that the remainder $R_{|\tau|+2}$ is exactly $\Pi_x \mathcal{I}(\tau)$ defined in (I.42)! Therefore, for $\mathrm{f} \equiv \tau$, (I.43) basically says

$$\Pi_x^\varepsilon[\mathcal{K}_+ \mathrm{f}(x)](y) = \Pi_x^\varepsilon \mathcal{I}(\tau)(y) + \sum_{|k|<|\tau|+2} D^k K_+ * w(x)\frac{(y-x)^k}{k!} + K_+ * (\mathcal{R}\mathrm{f} - w)(y) \tag{I.44}$$

Note that the last two terms in the last equality are all functions sufficiently smooth – they can be represented by abstract polynomials in the regularity structure. Replace them by the notation

$$\mathcal{J}(x)\tau = \sum_{|k|<|\tau|+2} D^k K_+ * (\Pi_x^\varepsilon \tau)(x)\frac{X^k}{k!},$$

$$(\mathcal{N}\mathrm{f})(x) = \sum_{|k|<|\tau|+2} [D^k K_+ * (\mathcal{R}\mathrm{f} - \Pi_x^\varepsilon \mathrm{f}(x))](x)\frac{X^k}{k!}.$$

Then (I.44) becomes

$$\Pi_x^\varepsilon \left[\mathcal{K}_+ \mathrm{f}(x)\right] = \Pi_x^\varepsilon \left[\mathcal{I}(\mathrm{f}(x)) + \mathcal{J}(x)\mathrm{f}(x) + (\mathcal{N}\mathrm{f})(x)\right].$$

This suggests us to define

$$\mathcal{K}_+\mathrm{f}(x) = \mathcal{I}\mathrm{f}(x) + \mathcal{J}(x)\mathrm{f}(x) + (\mathcal{N}\mathrm{f})(x) \tag{I.45}$$

for all modelled distribution f in order to have (I.43). Indeed, (I.45) turns out to be the right definition for our desired abstract representative of K_+.

On the other hand, the smooth kernel K_- is easy to deal with: one just defines

$$\mathcal{K}_-\mathrm{f}(x) = \sum D^k (K_- * \Pi_x^\varepsilon \mathrm{f}(x))(x)\frac{X^k}{k!}$$

and one naturally has $\mathcal{R}\mathcal{K}_-\mathrm{f} = K_- * \mathcal{R}\mathrm{f}$. Finally, we pose $\mathcal{K} = \mathcal{K}_+ + \mathcal{K}_-$.

Let us summarize the discussion and give the following definition.

Definition I.36. A model (Π, Γ) is said to be *admissible*, if equations (I.40) and (I.42) are satisfied. The space of admissible models[5] is denoted by $\mathscr{M}$ which is a closed space under the topology induced by Definition I.28.

Remark I.37. Note that for admissibility we do not assume (I.41). This is because (I.41) will be lost during renormalization as we will see below.

Provided that $(\Pi^\varepsilon, \Gamma^\varepsilon)$ is admissible for the kernel K_+, there exists an abstract convolution kernel $\mathcal{K}$ (defined by (I.45)) with the properties:

- For all $\mathrm{f} \in \mathscr{D}^\gamma$, it holds that

  $$\mathcal{R}(\mathcal{K}\mathrm{f}) = K_a * \mathcal{R}f$$

  where $\mathcal{R}$ is the reconstruction operator.

- $\mathcal{K}$ is a continuous linear map from $\mathscr{D}^\gamma$ to $\mathscr{D}^{\gamma+2}$ for all $\gamma \in A\backslash\mathbb{N}$. In particular,

  $$|\!|\!|\mathcal{K}\mathrm{f}|\!|\!|_{\gamma+2} \lesssim |\!|\!|\mathrm{f}|\!|\!|_\gamma$$

  for all $\mathrm{f} \in \mathscr{D}^\gamma$.

[5] One should not confuse the space of modelled distributions with the model space.

For complete proofs of these facts, we refer the readers to the original paper of Hairer [Hai14, Sec. 5].

Remark I.38. With the Besov-like space $\mathscr{D}^{\gamma}_{p,q}$ of modelled distributions defined in Remark I.35, one can actually have

$$\|\mathcal{K}\mathrm{f}\|_{\mathscr{D}^{\gamma+2}_{p,q}} \lesssim \|\mathrm{f}\|_{\mathscr{D}^{\gamma}_{p,q}}.$$

Moreover, by using the embedding property $\mathscr{D}^{\gamma+2}_{p,q} \hookrightarrow \mathscr{D}^{\gamma+2-\kappa}_{p,q}$ for all $\kappa > 0$, we can deduce

$$\|\mathcal{K}\mathrm{f}\|_{\mathscr{D}^{\gamma+2-\kappa}_{p,q}} \lesssim a^{-\kappa/2}\|\mathrm{f}\|_{\mathscr{D}^{\gamma}_{p,q}}.$$

With the prefactor $a^{-\kappa/2}$, we can take $a > 0$ large enough so that the map $\mathrm{u} \mapsto \mathcal{K}\mathrm{g} - \mathcal{K}(\mathrm{u}\Xi)$ is a contraction on $\mathscr{D}^{\frac{d}{2}+2\kappa}_{2,2}$, allowing us to obtain the desired fixed point for (I.31). Upon applying the reconstruction operator $\mathcal{R}^{\varepsilon}$ associated to the model $(\Pi^{\varepsilon}, \Gamma^{\varepsilon})$ defined by (I.40), (I.41) and (I.42), we get the solution to (I.29) before renormalization, i.e., with $C_{\varepsilon} = 0$.

Remark I.39. Let us point out that the notion of abstract convolution kernel depends on the underlying model. As a consequence, we should have denoted the kernel by $\mathcal{K}^{\varepsilon}$ to mark its dependency on $(\Pi^{\varepsilon}, \Gamma^{\varepsilon})$. We did not do so in order to avoid distracting notations. However, we should stress that this dependency on model is continuous: it implies that the fixed point u_{ε} to (I.31) is a continuous function of $(\Pi^{\varepsilon}, \Gamma^{\varepsilon})$. This fact is important because we would like to pass the models to limit in the end.

I.4.4 Renormalization

With the notion of admissible models and abstract convolution kernel, we have almost completed the picture given in I.1. The final step to make is to let ε shrink to 0: if the admissible model $(\Pi^{\varepsilon}, \Gamma^{\varepsilon})$ defined by (I.40), (I.41) and (I.42) converge, we would automatically have the convergence of fixed point, allowing to give a sense to the solution of (I.28) (see Remarks I.38 and I.39).

Unfortunately, this is not true in general: Take the symbol $\Xi\mathcal{I}(\Xi) \in \mathcal{T}_{\text{Anderson}}$ for example, one has

$$\Pi^{\varepsilon}_x[\Xi\mathcal{I}(\Xi)](y) = (\Pi^{\varepsilon}_x\Xi)(y) \cdot (\Pi^{\varepsilon}_x\mathcal{I}(\Xi))(y) = \xi_{\varepsilon}(y) \cdot [K_a * \xi_{\varepsilon}(y) - K_a * \xi_{\varepsilon}(x)]$$

which has no chance to converge since the only possible limit for the first term is $\xi \cdot (K_a * \xi)$, which does not satisfy Young's criterion of product between distributions.

To remedy it, we are going to modify the model $(\Pi^{\varepsilon}, \Gamma^{\varepsilon})$, resulting in another admissible model $(\hat{\Pi}^{\varepsilon}, \hat{\Gamma}^{\varepsilon})$, before passing to limit: this is the process of renormalization. We are going to give up the condition I.41, allowing us to *twist the product.* Let us look again at the symbol $\Xi\mathcal{I}(\Xi)$ and pose $\hat{\Pi}^{\varepsilon}$ such that

$$\hat{\Pi}^{\varepsilon}_x[\Xi\mathcal{I}(\Xi)] = \Pi^{\varepsilon}_x[\Xi\mathcal{I}(\Xi) - C_{\varepsilon}\mathbf{1}]$$

where Π^{ε}_x is as before. It turns out that one can choose a suitable constant C_{ε} so that the distribution $\hat{\Pi}^{\varepsilon}_x[\Xi\mathcal{I}(\Xi)]$ converges. Indeed, by the Wiener chaos decomposition [Nua06], for every test function $\eta \in C_c^\infty$, the random variable $\langle \xi_\varepsilon \cdot (K_a * \xi_\varepsilon), \eta\rangle$ can be written as

$$\begin{aligned}\langle \xi_\varepsilon \cdot (K_a * \xi_\varepsilon), \eta\rangle &= \int_{\mathbb{R}^d} \mathbb{E}[\xi_\varepsilon(x) K_a * \xi_\varepsilon(x)]\eta(x)\,\mathrm{d}x \\ &\quad + \iint \left(\int_{\mathbb{R}^d} \eta(x) K_{a,\varepsilon}(x-z) K_{a,\varepsilon}(x-z')\,\mathrm{d}x\right)\xi(\mathrm{d}z)\xi(\mathrm{d}z'),\end{aligned}$$

with $K_{a,\varepsilon} = K_a * \rho_\varepsilon$. Note that the second term on the right-hand side does converge as $\varepsilon \to 0$ in the second order homogeneous Wiener chaos. On the other hand, since the law of white noise is stationary, the expectation $\mathbb{E}[\xi_\varepsilon(x) K_a * \xi_\varepsilon(x)]$ is a constant value equal to

$$\iint_{(\mathbb{R}^d)^2} \rho_\varepsilon(x) K_a(x-y) \rho_\varepsilon(y)\,\mathrm{d}x\,\mathrm{d}y, \tag{I.46}$$

which will diverge as $\varepsilon \to 0$ due to the singularity of K_a on the diagonal. Therefore, if we set C_ε to the above value, then $\hat{\Pi}_x^\varepsilon[\Xi\mathcal{I}(\Xi)]$ will have a well-defined limit when $\varepsilon \to 0$!

In light of this, what we need to do is to single out all *trouble makers*, in particular those symbols with negative homogeneity, and make necessary modifications. Let $\mathcal{T}_0 \subset \mathcal{T}_{\text{Anderson}}$ be the vector space generated by the symbols:

$$\mathcal{T}_0 = \begin{cases} \operatorname{span}\{\Xi, \Xi\mathcal{I}(\Xi), X_i\Xi, \mathbf{1}\}, & d = 2, \\ \\ \operatorname{span}\{\Xi, \Xi\mathcal{I}(\Xi), \Xi\mathcal{I}(\Xi\mathcal{I}(\Xi)), X_i\Xi, \Xi\mathcal{I}(\Xi\mathcal{I}(\Xi\mathcal{I}(\Xi))), \Xi\mathcal{I}(\Xi X_i), \mathbf{1}\}, & d = 3, \end{cases}$$

What we mean by modification is expressed through a linear map $M : \mathcal{T}_0 \to \mathcal{T}_0$ satisfying

$$M\mathcal{I} = \mathcal{I}M, \quad M(\tau_1\tau_2) = M\tau_1 \cdot M\tau_2,$$

and also some technical properties ensuring that M "respects the algebraic structure" introduced in section I.4.2. Given such a M, it is possible to find associated model $(\hat{\Pi}^\varepsilon, \hat{\Gamma}^\varepsilon)$ such that $\hat{\Pi}_x^\varepsilon = \Pi_x^\varepsilon M$. The collection of all these possible maps M forms a group, called the *renormalization group* (see [Hai14, Section 8.3] for details). Now we only need to choose a suitable sequence (M_ε) in the renormalization group such that $(\hat{\Pi}^\varepsilon, \hat{\Gamma}^\varepsilon)$ converges.

It turns out that for $\mathcal{T}_{\text{Anderson}}$ in dimension $d = 2$, the only modification we need to make is on $\Xi\mathcal{I}(\Xi)$: Let C_ε be the quantity (I.46), define

$$M_\varepsilon[\Xi\mathcal{I}(\Xi)] = \Xi\mathcal{I}(\Xi) - C_\varepsilon\mathbf{1}, \quad M_\varepsilon\tau = \tau \text{ for all symbols } \tau \neq \Xi\mathcal{I}(\Xi)$$

and extends linearly to all $\mathcal{T}_0$. Then there exists a sequence of models $(\hat{\Pi}^\varepsilon, \hat{\Gamma}^\varepsilon)$ such that $\hat{\Pi}_x^\varepsilon = \Pi_x^\varepsilon M_\varepsilon$ and that $(\hat{\Pi}^\varepsilon, \hat{\Gamma}^\varepsilon)$ converges to some $(\hat{\Pi}, \hat{\Gamma})$ in $\mathscr{M}$. Let u (resp. u_ε) be the fixed point of (I.31) associated to the renormalized model $(\hat{\Pi}, \hat{\Gamma})$ (resp. $(\hat{\Pi}^\varepsilon, \hat{\Gamma}^\varepsilon)$) and let $\mathcal{R}$ (resp. $\mathcal{R}_\varepsilon$) be the reconstruction operator associated to $(\hat{\Pi}, \hat{\Gamma})$ (resp. $(\hat{\Pi}^\varepsilon, \hat{\Gamma}^\varepsilon)$). Then, by Theorem I.33 (and Remark I.35), $u = \mathcal{R}\mathrm{u}$ is the *renormalized solution* to the equation I.28 that we are looking for; moreover, by continuity with respect to models, we deduce $u_\varepsilon \to u$ in some Sobolev space.

Now a natural question arises: Does the solution yielded by the new model $(\hat{\Pi}, \hat{\Gamma})$ still satisfy a PDE related to the original equation after all? To answer the question, let us recall that the abstract fixed point $\mathrm{u} \in \mathscr{D}^\gamma$ takes the form (with $\gamma = \frac{d}{2} + 2\kappa$)

$$\mathrm{u}(x) = u_\varepsilon(x)\mathbf{1} + u^{\mathcal{I}(\Xi)}(x)\mathcal{I}(\Xi) + u^{X_1}(x)X_1 + u^{X_2}(x)X_2, \quad x \in \mathbb{T}^2,$$

for some coefficients $u^{\mathcal{I}(\Xi)}(x), u^{X_1}(x), u^{X_2}(x) \in \mathbb{R}$. Observe that

$$M_\varepsilon \mathrm{u}(x) = (u_\varepsilon(x) - C_\varepsilon)\mathbf{1} + u^{\mathcal{I}(\Xi)}(x)\mathcal{I}(\Xi) + u^{X_1}(x)X_1 + u^{X_2}(x)X_2.$$

As $\Pi_x\tau(x) = 0$ for all τ with strictly positive homogeneity and all $x \in \mathbb{T}^2$, it follows that

$$\hat{\Pi}^\varepsilon_x[\Xi\mathrm{u}(x)](x) = \Pi^\varepsilon_x M_\varepsilon[\Xi\mathrm{u}(x)](x) = \xi_\varepsilon(x)(u_\varepsilon(x) - C_\varepsilon).$$

Apply $\mathcal{R}_\varepsilon$ to (I.31) and use the fact that $\mathcal{R}_\varepsilon\mathrm{u}(x) = \hat{\Pi}^\varepsilon_x[\mathrm{u}(x)](x)$ and $\mathcal{R}_\varepsilon\mathcal{K}\mathrm{u} = K_a * \mathcal{R}_\varepsilon\mathrm{u}$, it follows that

$$u_\varepsilon = \mathcal{R}_\varepsilon\mathrm{u} = K_a * g - K_a * \mathcal{R}_\varepsilon(\Xi\mathrm{u}) = K_a * g - K_a * [\xi_\varepsilon(u_\varepsilon - C_\varepsilon)].$$

Equivalently, u_ε solves (I.30). By continuity, the limit u of u_ε solves the formal equation

$$-\Delta u + au + (\xi + \infty)u = 0$$

in renormalization sense.

I.5 OVERVIEW OF SUBSEQUENT CHAPTERS

The following chapters of the present thesis are dedicated to two general objectives:

1. To construct $\mathcal{H}$ represented in (I.1) under different settings as a *self-adjoint operator-valued* random variable.
2. To understand the (a priori random) spectrum of $\mathcal{H}$.

Let us recall the diagram in Figure I.1, which can be summarized into the map in Figure I.2. As mentioned previously, the maps (b) and (c) are continuous: The continuity of (b)

$$\begin{array}{ccccccc} \Omega & \xrightarrow{(a)} & \mathscr{M} & \xrightarrow{(b)} & \text{Solution to (I.11) or (I.12)} & \xrightarrow{(c)} & \text{Operator} \\ \xi & \mapsto & (\Pi, \Gamma) & \mapsto & u & \mapsto & \mathcal{H} \end{array}$$

Figure I.2: Summary of Figure I.1.

is a consequence of the continuity of reconstruction operator (section I.4.1) and that of the fixed point with respect to the models (section I.4.3); The continuity of (c) can be proved by (some generalization of) the ideas from section I.3.2. On the other hand, the map (a) is only measurable. From these observations, the objective 1. can be achieved if we fulfill the program $(a), (b), (c)$.

To tackle the objective 2., we regard an admissible model $Z = (\Pi, \Gamma) \in \mathscr{M}$ as a random variable defined on the canonical probability space Ω of the white noise ξ. By studying the law of the model, one can hope to recover some spectral properties of the operator $\mathcal{H}$ by the continuity of maps (b) and (c).

In the following, we outline the results presented in the subsequent chapters, obtained using this general reasoning.

chapter II: asymptotic of anderson eigenvalues

This chapter is based on the joint work with C. Labbé [HL22], published in *Stochastics and Partial Differential Equations: Analysis and Computations*, Volume 11, Issue 3, September 2023.

We study the continuous Anderson Hamiltonian $\mathcal{H}_L$, i.e., with $\mathbf{A} = 0$ and $V = \xi$ in (I.1), defined on $L^2((-L/2, L/2)^d)$ for $d = 1, 2, 3$. Its construction is provided by [Lab19], which combined the idea of mild equation (section I.3.2, Construction I) and the solution by regularity structure outlined in section I.4. This argument affirms that $\mathcal{H}_L$ is a self-adjoint operator-valued random variable, and a.s. the operator $\mathcal{H}_L$ has compact resolvent. Therefore, it admits purely discrete spectrum:

$$\lambda_{1,L} \le \lambda_{2,L} \le \lambda_{3,L} \le \cdots .$$

The main result of this work concerns the almost sure asymptotic of eigenvalues $\lambda_{n,L}$ as $L \to \infty$.

Theorem I.40 (Theorem II.1). *Fix $d \in \{1, 2, 3\}$ and $n \ge 1$. It holds almost surely that*

$$\lambda_{n,L} = -(C_d \log L)^{\frac{1}{2-d/2}}(1 + o(1)), \quad L \to \infty. \tag{I.47}$$

Moreover, the constant C_d is explicit: it depends only on the dimension d and is closely related to the optimal constant of the Gagliardo-Nirenberg inequality.

In dimension $d = 1$, the constant C_1 equals $\frac{8}{3}$, leading to the asymptotic $\lambda_{n,L} \sim (\frac{8}{3} \log L)^{2/3}$, which matches the result obtained by McKean [McK94] that was presented in section I.3.4. In dimension $d = 2$, the asymptotic reads $\lambda_{n,L} \sim C_2 \log L$ with

$$C_2 = \sup_{f \in H^1 \setminus \{0\}} \frac{\|f\|_{L^4}^4}{\|\nabla f\|_{L^2}^2 \, \|f\|_{L^2}^2},$$

which matches the result by Chouk and Van Zuijlen [Cv21]. In dimension $d = 3$, our result is new.

Let us outline the key argument for the proof. We first need some scaling results which relate the eigenvalues $\lambda_{n,L}$ to the operator associated to noise with rescaled density

$\beta\xi$, $\beta > 0$. Denote by $Z(\beta)$ the renormalized model associated to $\beta\xi$. Recall that $(Z(\beta))_{\beta>0}$ is a family of random variables taking values in $\mathscr{M}$, the space of admissible models. As $\beta \to 0$, Hairer and Weber [HW15] proved that the law of $Z(\beta)$ satisfies a large deviation principle (LDP) with rate β^2 and some rate function $\mathscr{I}$. Since the operator $\mathcal{H}_L(\beta) = -\Delta + \beta\xi$ is a continuous function of $Z(\beta)$ (due to the continuity of the map $(b), (c)$ in Figure I.2), by the contraction principle in large deviation theory, we are able to deduce a large deviation estimate which basically says $\lambda_{n,L}$ "almost" verifies a LDP when $L \to \infty$. This finally leads to the asymptotic (I.47) and the identification of the prefactor C_d.

chapter III: spectrum of continuous anderson hamiltonian on the whole space

This chapter is based on the joint work with C. Labbé [HL24], submitted to *Probability and Mathematical Physics*.

We study the continuous Anderson Hamiltonian $\mathcal{H}$ on $L^2(\mathbb{R}^d)$ i.e., with $\mathbf{A} = 0$ and $V = \xi$ in (I.1). The construction of such an operator on full $\mathbb{R}^d$ is more difficult than the finite volume case on $(-L/2, L/2)^d$ since $\mathcal{H}$ is not expected to semibounded: indeed, the asymptotics of the smallest eigenvalues in the infinite volume limit presented right above indicate that the bottom of the spectrum of the infinite volume operator should be $-\infty$. Consequently, the ideas of weak formulation and the mild formulation presented in section I.3.2 do not apply (because the quadratic form is not bounded from below, and one does expect to find a sufficiently large a for which (I.11) is solvable). See also section I.3.3 for a related discussion.

Instead we follow the idea of “Construction III” from section I.3.2. Recall that the starting point therein is to construct the semigroup (that is, the parabolic initial value problem), and then to extract its generator.

The construction of the parabolic PDE is not standard since the potential is rough. In addition, this semigroup is not bounded anymore in our setting so that the extraction of the generator from the semigroup is not immediate.

Our idea is to solve the Cauchy problem

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x) - \xi(x)u(t,x), & t > 0, x \in \mathbb{R}^d \\ u(t,0) = g \in L^2, & t = 0 \end{cases} \tag{I.48}$$

and prove that the solution verifies the condition in Theorem III.3; one then can extract the generator $\mathcal{H}$, which defines the desired Anderson Hamiltonian. Indeed, the solution theory to (I.48) being available in [HL15] for dimension 2 and in [HL18] for dimension 3, we essentially only need to adapt their setting to L^2 and prove the continuity of the map $t \mapsto u(t,\cdot)$ as a function-valued map in a suitable sense.

Moreover, as the white noise ξ is ergodic with respect to the translation in $\mathbb{R}^d$, one can show that the self-adjoint operator $\mathcal{H}$ so-constructed has deterministic spectrum (Proposition I.21), i.e., there exists a non-random set $\Sigma \subset \mathbb{R}$ such that $\sigma(\mathcal{H}) = \Sigma$ almost surely. The second part of this chapter concerns the identification of Σ, which we show equals the whole real line $\mathbb{R}$. We summarize the result below:

Theorem I.41 (Theorems III.1 and III.2)**.** *For $d = 2, 3$, the continuous Anderson Hamiltonian $\mathcal{H}$ is well-defined over a dense domain $\mathcal{D}(\mathcal{H}) \subset L^2(\mathbb{R}^d)$. Moreover, $\mathcal{H}$ is ergodic and its spectrum coincides with $\mathbb{R}$ almost surely.*

The above result in dimension $d = 2$ has been obtained recently by Ueki [Uek23] using the sophisticated theory of paracontrolled distribution based on heat kernels; our argument for $d = 2$ appears to be much more elementary. On the other hand, the result for dimension $d = 3$ is new.

The key heuristics to show $\Sigma = \mathbb{R}$ follows from the following observation due to Kotani [Kot85], see also [CL90, Corollary V.2.3.]:
Suppose Ω is a probability space on which $\mathcal{H} : \xi \mapsto \mathcal{H}^\xi$ is a self-adjoint operator-valued random variable. Fix any two probability measure $\mathbb{P}_1, \mathbb{P}_2$ on Ω for which $\mathcal{H}$ is ergodic, and let Σ_1, Σ_2 be the deterministic spectrum of $\mathcal{H}$ under $\mathbb{P}_1$, $\mathbb{P}_2$, respectively. Assume that Ω is a Polish space on which the following continuity property for spectral measure holds: for all $\varphi \in L^2$

$$\xi_n \to \xi \text{ in } \Omega \implies \left\langle E^{\xi_n}(\cdot)\varphi, \varphi \right\rangle \text{ converges weakly to } \left\langle E^{\xi}(\cdot)\varphi, \varphi \right\rangle, \tag{I.49}$$

where E^{ξ_n}, E^ξ denote the projection-valued measure of the operator $\mathcal{H}^{\xi_n}$, $\mathcal{H}^\xi$, respectively. Then one has

$$\operatorname{supp} \mathbb{P}_1 \subset \operatorname{supp} \mathbb{P}_2 \implies \Sigma_1 \subset \Sigma_2.$$

Using this observation, the idea is to set $\mathbb{P}_2$ to be law of the Gaussian white noise and $\mathbb{P}_1$ the law of the regularized noise ξ_ε, for any fixed $\varepsilon > 0$. Obviously, one has $\operatorname{supp} \mathbb{P}_1 \subset \operatorname{supp} \mathbb{P}_2$ and it is not hard to show that $\Sigma_1 = \mathbb{R}$ in this case, which would imply $\Sigma_2 = \mathbb{R}$ if (I.49) was true. Unfortunately, the property (I.49) fails in the singular case of white noise due to the fact that the map (a) in Figure I.2 is only measurable.

To restore the continuity, we focus on the space of admissible models $\mathscr{M}$ instead. Let $Z_\varepsilon, Z_\varepsilon \in \mathscr{M}$ be the models associated to ξ_ε, ξ, respectively. If we can show that the support of the laws of Z_ε and Z verify

$$\operatorname{supp} Z_\varepsilon \subset \operatorname{supp} Z, \tag{I.50}$$

then one can deduce $\Sigma \supset \Sigma_\varepsilon = \mathbb{R}$ from Kotani's theorem, since the projection-valued measure is continuous with respect to the model (maps (b) and (c) in Figure I.2).

In Chapter III, we give a complete argument for (I.50) in dimension 2 and we rely on the highly non-trivial result of Hairer and Schönbauer [HS21] for dimension 3.

Chapter IV: Landau Hamiltonian with white noise potential on whole space $\mathbb{R}^2$

This chapter is based on an ongoing independent work.

We study a construction (without using any singular SPDE theory) of Landau Hamiltonian $\mathcal{H}_{\mathbf{A}}$ on $L^2(\mathbb{R}^2)$ with uniform magnetic field and perturbed by white noise potential. That is, $\mathcal{H}_{\mathbf{A}}$ takes the form (I.1) with $\mathbf{A}(x,y) = \frac{B}{2}(-y,x)$, $B > 0$ and $V = \xi$. The key argument is to use the commutator theorem of Faris-Lavine (Proposition IV.16), an idea originated from Ugurcan [Ugu22] concerning the continuous Anderson Hamiltonian.

The desired result is the following:

Theorem I.42 (Theorem IV.1)**.** *The Landau Hamiltonian $\mathcal{H}_{\mathbf{A}}$ with uniform magnetic field perturbed by white noise potential on $L^2(\mathbb{R}^2)$ is well-defined and self-adjoint over a dense domain.*

The idea of proof is to decompose the white noise into the sum

$$\xi = \xi_- + \xi_+$$

of the singular part ξ_-, which is *globally* $(-1)^-$-Hölder a.s., and the unbounded part ξ_+, which is continuous and satisfies $\xi_+(x) \geq -C(1+|x|^2)$ for $0 < C < \infty$ almost surely. With the decompostion at hand, we can reason in two steps:

1. Firstly, construct the singular operator $T = (i\nabla + \mathbf{A})^2 + \xi_-$. Here, the weak formulation (section I.3.2, Construction II) combined with the exponential transformation trick introduced in [HL15] allows to realize T as a self-adjoint random operator.

2. Secondly, define $\mathcal{H}_{\mathbf{A}} = T + \xi_+$ and use Faris-Lavine Theorem IV.16 to show that $\mathcal{H}_{\mathbf{A}}$ is essentially self-adjoint on the domain $\mathscr{C} = \{u \in \mathcal{D}(T) : |x|^2 u \in L^2\}$.

However, in the writing of Chapter IV, we realize that there is a gap in the second step, which is already present in the original paper of Ugurcan [Ugu22]. Indeed, to apply Faris-Lavine, one defines the auxiliary operator

$$N = T + \xi_+ + C(1+|x|)^2.$$

One aims to show that N is a positive, (essentially) self-adjoint operator over $\mathscr{C}$ such that the commutator estimate holds:

$$\pm i[\mathcal{H}_{\mathbf{A}}, N] \lesssim N$$

(interpreted in the sense of quadratic form over $\mathscr{C}$). [Ugu22] argued that N by itself is self-adjoint on $\mathscr{C}$ and the commutator estimate holds true, whence H is essentially self-adjoint. The gap lies in the reasoning for the self-adjointness of N: Ugurcan showed that N is symmetric, closed and positive over $\mathscr{C}$, and concluded that N has a real resolvent value and therefore is self-adjoint – this conclusion is fallacious since these properties (symmetric, closed and positive) do not imply the existence of a resolvent value. A classical example is provided by the one-dimensional Laplacian on the interval $(0,1)$ with the domain $H_0^2((0,1))$: this operator is symmetric, closed and positive; however its spectrum is the whole complex plane $\mathbb{C}$.

Indeed, the symmetry and positiveness of N do imply the existence of a self-adjoint extension by Friedrichs' theorem I.13. However, this existence does not necessarily coincides with the closure of N over $\mathscr{C}$; in fact, we have a priori no control on the domain of Friedrichs' extension and it could be much larger than the original set $\mathscr{C}$. This makes the assumptions of Faris-Lavine theorem fail.

Despite the gap in the construction of the operator $\mathcal{H}_{\mathbf{A}}$ on full space, we do obtain some partial results for the operator $\mathcal{H}_{\mathbf{A},L}$ constructed on bounded box $(-L/2, L/2)^d$.

Theorem I.43 (Theorem IV.2)**.** *The Landau Hamiltonian $\mathcal{H}_{\mathbf{A},L}$ with uniform magnetic field and Gaussian white noise potential on $(-L/2, L/2)^2$ with Dirichlet boundary condition has compact resolvent and its eigenvalues admit the same asymptotics as* (I.47) *in $d = 2$.*

Let us finally mention that it is possible to construct $\mathcal{H}_{\mathbf{A}}$ using Klein-Landau theorem as in Chapter III, but it requires the theory of regularity structures. We do not present the details but we intend to implement this in the future.

Appendix: strong resolvent convergence from finite volume to full space

We include a simple new result in the appendix, addressing the convergence of Anderson Hamiltonians defined on finite volume to that defined on the full space, a notion sometimes called the *thermaldynamic limit.* We consider dimensions $d \in \{2, 3\}$.

Theorem I.44 (Theorem A.1)**.** *Let $\mathcal{H}$ be the continuous Anderson Hamiltonian on $L^2(\mathbb{R}^d)$ and let $\mathcal{H}_L$ be the its counterpart on $L^2((-L/2, L/2)^d)$. Then, $\mathcal{H}_L$ converges to $\mathcal{H}$ in strong resolvent sense as $L \to \infty$, in probability.*

Chapter II

Asymptotic of the smallest eigenvalues of the continuous Anderson Hamiltonian in $d \leq 3$

This chapter is based on the article [HL22], published in *Stochastics and Partial Differential Equations: Analysis and Computations*, Volume 11, Issue 3, September 2023.

SUMMARY OF THE PRESENT CHAPTER

II.1 INTRODUCTION

Given a white noise ξ on $\mathbb{R}^d$, we consider the truncated continuous Anderson Hamiltonian

$$\mathcal{H}_L := -\Delta + \xi, \quad \text{on } Q_L := (-L/2, L/2)^d,$$

where Δ is the continuous Laplacian, boundary conditions are taken to be homogeneous Dirichlet and the dimension d is either 1, 2 or 3.

This operator belongs to the class of random Schrödinger operators. The particularity of the present setting is the singularity of the white noise potential, which is only distribution-valued. In dimension 1, the operator $\mathcal{H}_L$ can be defined with standard tools and rather complete results are available on the asymptotic behaviour as $L \to \infty$ of its eigenvalues and eigenfunctions, see [McK94, DL20, DL23, DL24b].

On the other hand, the mere definition of the operator in dimensions 2 and 3 is a priori unclear. Indeed, the regularity of white noise is too low for the operator to be defined by classical arguments, and it actually needs to be *renormalised* by infinite constants. New techniques in the field of stochastic PDEs have provided the appropriate tools to carry out such a construction. Building on the paracontrolled calculus of Gubinelli, Imkeller and Perkowski [GIP15], Allez and Chouk [AC15] constructed $\mathcal{H}_L$ in dimension 2 under periodic b.c. This construction was extended to dimension 3 under periodic b.c. in [GUZ20], under Dirichlet b.c. in dimension 2 in [Cv21] and to 2-dimensional manifolds in [Mou21]. On the other hand, a construction under periodic and Dirichlet b.c. and for any dimension $d \leq 3$ was presented in [Lab19] using the theory of regularity structures [Hai14]: in the present article we rely on this construction for convenience.

Let us provide a brief description of the aforementioned renormalisation procedure. Consider the operator $\mathcal{H}_{\varepsilon,L} = -\Delta + \xi_\varepsilon + C_\varepsilon$ associated with a regularized noise $\xi_\varepsilon = \xi * \rho_\varepsilon$, where ρ_ε is a smooth function that lives at scale ε. This operator is well-defined since ξ_ε is a smooth function. In the references above, it is shown that if one chooses properly the *renormalisation constant* C_ε, then $\mathcal{H}_{\varepsilon,L}$ converges in norm resolvent sense to some limit that we call $\mathcal{H}_L$. Note that, as $\varepsilon \downarrow 0$, C_ε diverges logarithmically in dimension 2 and polynomially in dimension 3. We refer the reader to section II.3 for further details.

In fine, these constructions yield a self-adjoint operator $\mathcal{H}_L$ on $L^2((-L/2, L/2)^d)$ with pure point spectrum bounded from below: we let $(\lambda_{k,L})_{k\geq 1}$ be its eigenvalues in non-decreasing order and $(\varphi_{k,L})_{k\geq 1}$ be the corresponding eigenfunctions normalised in L^2. In contrast with dimension 1, very little is known on the spectrum of $\mathcal{H}_L$: in dimension 2, the asymptotic behaviour as $L \to \infty$ of the smallest eigenvalues was derived in [Cv21] while the existence of a density of states was proven in [Mat21]; in dimension 3, essentially no result on the spectrum is available.

For later use, let us recall the Gagliardo-Nirenberg inequality - also referred to as Ladyzhenskaya's inequality

$$\|f\|_{L^4(\mathbb{R}^d)} \leq C \|\nabla f\|_{L^2(\mathbb{R}^d)}^{d/4} \|f\|_{L^2(\mathbb{R}^d)}^{1-d/4} , \tag{II.1}$$

and let κ_d be the associated optimal constant, that is

$$\kappa_d := \sup_{f \in H^1(\mathbb{R}^d)} \frac{\|f\|_{L^4(\mathbb{R}^d)}}{\|\nabla f\|_{L^2(\mathbb{R}^d)}^{d/4} \|f\|_{L^2(\mathbb{R}^d)}^{1-d/4}} . \tag{II.2}$$

The main result of this article is as follows.

Theorem II.1. *Fix $d \in \{1,2,3\}$ and $n \in \mathbb{N}$. Then almost surely*

$$\lambda_{n,L} \sim -\left(C_d \log L\right)^{\frac{1}{2-\frac{d}{2}}} \quad , \quad L \to \infty \,. \tag{II.3}$$

The constant C_d can be expressed in terms of the Gagliardo-Nirenberg constant through the relation

$$C_d = \frac{d^{1+\frac{d}{2}}(4-d)^{2-\frac{d}{2}}}{8}\, \kappa_d^4 \,. \tag{II.4}$$

Let us make some comments on this result. In dimension 1, the Gagliardo-Nirenberg constant is known to be $\kappa_1 = 3^{-1/8}$ and the result rewrites

$$\lambda_{n,L} \sim -\left(\frac{3}{8}\log L\right)^{2/3} \quad , \quad L \to \infty \,.$$

This asymptotic is already covered by more precise results [McK94, DL20], in which not only the asymptotic behaviour of $\lambda_{n,L}$ but also its fluctuations are derived, see the end of the introduction for more details. It can also be connected to a result of Chen [Che10] on the total-mass of the associated parabolic Anderson model.

In higher dimension the Gagliardo-Nirenberg constant κ_d is not explicit anymore. In dimension 2, the asymptotic is

$$\lambda_{n,L} \sim -\kappa_2^4 \log L \,, \quad L \to \infty \,,$$

and was recently established by Chouk and van Zuijlen [Cv21] (see also [GL22] for related results for smooth Gaussian noises). A minor improvement over their result is that our convergence holds almost surely over $L \to \infty$, and not only over sequences $L_k \to \infty$, see Remark II.4 for some explanations. In dimension 3, the asymptotic is

$$\lambda_{n,L} \sim -\frac{243}{64}\kappa_3^8 \left(\log L\right)^2 \quad , \quad L \to \infty \,,$$

and in that case, the result is new.

Our proof is carried out simultaneously in all dimensions $d \le 3$ in order to emphasise the dependence on d of the arguments and of the overall result. Let us point out that we follow the same strategy of proof as Chouk and van Zuijlen [Cv21] who covered the case of dimension 2. In fact, the proof essentially boils down to establishing a tail estimate on the principal eigenvalue: this is the content of the next (more general) result.

Theorem II.2. *Fix $\eta \in (0,1)$ and $n \ge 1$. There exist $\gamma_2 > \gamma_1 > 0$ and $x_0 > 0$ such that the following inequalities hold: for all $L \ge 1$ and all $x \ge x_0$ we have*

$$\exp\left[-\gamma_2 x^{\frac{d}{2}} e^{d\log L - (1-\eta)\rho x^{2-\frac{d}{2}}}\right] \le \mathbb{P}(\lambda_{n,L} \ge -x) \le \exp\left[-\gamma_1 x^{\frac{d}{2}} e^{d\log L - (1+\eta)\rho x^{2-\frac{d}{2}}}\right] \tag{II.5}$$

with $\rho = d/C_d$.

Observe that in the limit $L \to \infty$ and for small η (take $\eta = 0$ for simplicity), the leftmost and rightmost functions in (II.5) pass abruptly from 0 to 1 around the critical value $x_c = (C_d \log L)^{1/(2-d/2)}$ where the exponent $d \log L - \rho x^{2-d/2}$ vanishes. This implies that the distribution function $x \mapsto \mathbb{P}(\lambda_{n,L} \geq -x)$ is close to 0 for $x \ll x_c$ and close to 1 for $x \gg x_c$, and therefore that the distribution of $-\lambda_{n,L}$ concentrates near this critical value. Given this result, the derivation of Theorem II.1 is relatively elementary.

We conclude this introduction with some conjectures. Let a_L be the unique solution to the equation

$$\frac{d}{2}\log a_L + d\log L - \rho a_L^{2-d/2} = 0 \;,$$

and set

$$b_L := \frac{C_d}{d(2-\frac{d}{2})a_L^{1-\frac{d}{2}}} \;.$$

Note that the asymptotic expansion of a_L is given by

$$a_L^{2-d/2} := (C_d \log L)\left[1 + \frac{1}{4-d}\frac{\log\log L}{\log L} + o\left(\frac{\log\log L}{\log L}\right)\right]. \tag{II.6}$$

Conjecture 1. *Take $d \in \{1,2,3\}$. The point process $\left(\frac{\lambda_{n,L}+a_L}{b_L}\right)_{n\geq 1}$ converges in law as $L \to \infty$ to a Poisson point process on $\mathbb{R}$ of intensity $e^x dx$. In particular, the r.v.*

$$-\frac{\lambda_{1,L}+a_L}{b_L}$$

converges in law to a Gumbel random variable.

Our second conjecture concerns the asymptotic behaviour of the eigenfunctions near their maxima. We let $U_{n,L} \in [-L/2, L/2]^d$ be the point where $|\varphi_{n,L}|$ reaches its global maximum. Let Q be the unique radial positive solution on $\mathbb{R}^d$ of

$$-\Delta Q - Q^3 = -Q \;.$$

It is known that - up to translations, dilatations and rescalings - Q is the unique optimiser of the Gagliardo-Nirenberg inequality (II.1), see [Fra14] and references therein. One can deduce from [Lew10, Sec.5] that $\|Q\|_{L^4}^4 = \frac{2d}{C_d}$.

Conjecture 2. *Take $d \in \{1,2,3\}$. For any $n \geq 1$, the following convergence holds in probability as $L \to \infty$*

$$\left(\frac{1}{a_L^{d/4}}|\varphi_{n,L}|\left(U_{n,L} + \frac{x}{\sqrt{a_L}}\right), x \in \mathbb{R}^d\right) \Rightarrow \psi_* \;,$$

$$\left(\frac{1}{a_L}\xi\left(U_{n,L} + \frac{x}{\sqrt{a_L}}\right), x \in \mathbb{R}^d\right) \Rightarrow -\frac{{\psi_*}^2}{\|\psi_*\|_{L^4(\mathbb{R}^d)}^2}\sqrt{\frac{2d}{C_d}} \;,$$

with $\psi_*(x) = Q(x)/\|Q\|_{L^2}$.

In Conjecture 2 the first convergence holds in a space of distributions; the abuse of notation regarding the scaling on ξ shall be interpreted in the distributional sense, i.e. passing the scaling operations to test functions.
In dimension 1, these two conjectures were actually proven by Dumaz and Labbé [DL20] (the convergence to a Gumbel r.v. was proven earlier by McKean [McK94]). In that case, we have

$$a_L \sim \left(\frac{3}{8}\log L\right)^{2/3}, \quad b_L = \frac{1}{4\sqrt{a_L}}, \quad \psi_* = \frac{1}{\sqrt{2}\cosh}, \quad \frac{{\psi_*}^2}{\|\psi_*\|^2_{L^4(\mathbb{R}^d)}}\sqrt{\frac{2d}{C_d}} = \frac{2}{\cosh^2}.$$

The present work is organized in the following way. In section II.2, we collect intermediate ingredients and provide the proofs of Theorem II.2 and Theorem II.1, together with the proofs of the ingredients that do not necessitate regularity structures. In section II.3, we prove some technical results on the Anderson Hamiltonian and present the proof of a large deviation estimate stated in section II.2. The Appendix collects some technical results.

II.2 PROOFS OF THE MAIN THEOREMS

We start this section by collecting some simple properties of the operator $\mathcal{H}_L$: actually, we will consider a more general framework where the spatial domain can be taken to be any given square box $Q \subset \mathbb{R}^d$, and where white noise comes with a prefactor $\beta > 0$, as it will be required later on. The second subsection presents a large deviation estimate for the main eigenvalue of $\mathcal{H}_L$ with a small noise ($\beta \downarrow 0$), along with some information on the associated rate function. These properties of the rate function and some results about the associated variational problem are proved in the third subsection. In the fourth subsection, we provide the proofs of the main theorems. Finally, the last subsection gives some heuristic explanations on Conjecture 2.

From now on, we call *box* any open bounded square box of $\mathbb{R}^d$ of side-length at least one and d is always assumed to lie in $\{1, 2, 3\}$.

II.2.1 SIMPLE PROPERTIES OF THE OPERATOR

Let Q be either a box in $\mathbb{R}^d$ or $\mathbb{R}^d$ itself, take $V \in L^2(Q)$, and define the operator

$$\mathcal{H}(Q, V) := -\Delta + V, \quad \text{on } Q,$$

endowed with Dirichlet b.c. It is well-known [Lew22, Hel13] that for $d \leq 3$, this operator is self-adjoint and bounded below. When Q is bounded, its spectrum is discrete and we denote its eigenvalues in non-decreasing order by $(\lambda_n(Q, V))_{n\geq 1}$. When $Q = \mathbb{R}^d$, we let $\lambda_1(Q, V)$ be the infimum of its spectrum.

The next result constructs the operator as a limit of regularised versions in the case where V is white noise. As recalled in the introduction, in dimension $d = 2, 3$ one needs to renormalise these regularised versions for the limit to exist.

Fix some even, smooth function ρ integrating to 1 and supported in the unit ball of $\mathbb{R}^d$. Set $\rho_\varepsilon := \varepsilon^{-d}\rho(\cdot/\varepsilon)$ for any $\varepsilon > 0$ and define $\xi_\varepsilon := \xi * \rho_\varepsilon$ for some white noise ξ. For any $\beta > 0$, we consider the renormalisation constant $C_\varepsilon(\beta)$ associated to the noise $\beta\xi_\varepsilon$: in order not to clutter the presentation, their precise expressions are provided in Appendix II.A. The proof of the following result is postponed to Subsection II.3.2 as it requires notions from regularity structures.

Proposition II.3. *Fix some parameter $\beta > 0$. There exists a sequence $\varepsilon_k \downarrow 0$ and an event $\Omega_0(\beta)$ of probability one on which the following holds:*

1. *For every box Q, the sequence $\mathcal{H}(Q, \beta\xi_{\varepsilon_k} + C_{\varepsilon_k}(\beta))$ converges in norm resolvent sense to a self-adjoint operator, denoted $\mathcal{H}(Q, \beta\xi)$ with a slight abuse of notation, with pure point spectrum bounded from below. We denote $(\lambda_n(Q, \beta\xi))_{n\geq 1}$ its eigenvalues in non-decreasing order.*
2. *For all $n \geq 1$ and all boxes $Q \subset Q'$, we have $\lambda_n(Q', \beta\xi) \leq \lambda_n(Q, \beta\xi)$.*
3. *For all $L \geq 1$, all $n \geq 1$ and all disjoint boxes $Q_{(1)}, \ldots, Q_{(n)} \subset Q$, we have*
$$\lambda_n(Q, \beta\xi) \leq \max_{1\leq i\leq n} \lambda_1(Q_{(i)}, \beta\xi) \ .$$

In the particular case $Q = Q_L = (-L/2, L/2)^d$ and $\beta = 1$, we abbreviate $\mathcal{H}(Q_L, \xi)$ into $\mathcal{H}_L$ and $\lambda_n(Q_L, \xi)$ into $\lambda_{n,L}$.

Remark II.4. Note that we construct the operator $\mathcal{H}(Q, \xi)$ *simultaneously* for all boxes Q, and this allows us to consider almost sure convergence of $\lambda_{n,L}$ as $L \to \infty$. This is to be compared with the construction in [Cv21] that holds up to a $\mathbb{P}$-null set that possibly depends on Q. The fundamental reason for our simultaneous construction is that all the stochastic objects (the so-called model in the theory of regularity structures) are constructed at once on the full space, while the dependence over the given box Q only goes through the deterministic weights chosen near the boundary of Q, see [Lab19].

Let us now collect some simple properties of these operators.

Proposition II.5 (Scaling, independence and invariance properties)**.**

1. *There exists a deterministic constant δ_β such that δ_β tends to 0 as $\beta \downarrow 0$ and such that for all $L \geq 1$, $\beta > 0$ and $n \geq 1$ the following equality in law holds*
$$\beta^2\lambda_n(Q_L, \xi) = \lambda_n(Q_{L/\beta}, \beta^{2-d/2}\xi) + \delta_\beta \ .$$
2. *For all disjoint boxes $Q_1, \ldots, Q_k$, the operators $\mathcal{H}(Q_1, \beta\xi), \ldots, \mathcal{H}(Q_k, \beta\xi)$ are independent.*
3. *For all boxes Q, Q' with the same side-length, $\mathcal{H}(Q, \beta\xi)$ and $\mathcal{H}(Q', \beta\xi)$ have the same law.*

Remark II.6. Properties 2. and 3. should be understood at the level of the resolvents of the operators. Note that the resolvents at stake are random variables taking values in the space of compact operators on $L^2(Q)$, which, equipped with the operator norm, is a separable Banach space.

Proof. The second and third properties are consequences of the independence and translation invariance properties of white noise, and of the construction of the operator $\mathcal{H}(Q, \beta\xi)$ as a limit of regularised operators. We concentrate on the first property. Let $\tilde{\xi}_\varepsilon(x) := \beta^2 \xi_\varepsilon(x\beta)$ for all $x \in \mathbb{R}^d$. Consider the self-adjoint operator

$$\tilde{\mathcal{H}}_\varepsilon := -\Delta + \tilde{\xi}_\varepsilon + C_{\varepsilon/\beta}(\beta^{2-\frac{d}{2}}) \ , \qquad \text{on } Q_{L/\beta} \ .$$

Let $\lambda_{n,L,\varepsilon}$ and $\varphi_{n,L,\varepsilon}$ be the n-th eigenvalue and eigenfunction of $-\Delta + \xi_\varepsilon + C_\varepsilon(1)$ on Q_L. A computation shows that

$$\tilde{\mathcal{H}}_\varepsilon \, \varphi_{n,L,\varepsilon}(\cdot\beta) = \Big(\beta^2 \lambda_{n,L,\varepsilon} + C_{\varepsilon/\beta}(\beta^{2-\frac{d}{2}}) - \beta^2 C_\varepsilon(1)\Big)\varphi_{n,L,\varepsilon}(\cdot\beta) \ .$$

We thus deduce that the n-th eigenvalue of $\tilde{\mathcal{H}}_\varepsilon$ coincides with $\beta^2 \lambda_{n,L,\varepsilon} + C_{\varepsilon/\beta}(\beta^{2-\frac{d}{2}}) - \beta^2 C_\varepsilon(1)$.
It turns out that $\tilde{\xi}_\varepsilon$ has the same law as $\beta^{2-\frac{d}{2}} \xi_{\varepsilon/\beta}$, and therefore $\tilde{\mathcal{H}}_\varepsilon$ has the same law as $\mathcal{H}(Q_{L/\beta}, \beta^{2-\frac{d}{2}} \xi_{\varepsilon/\beta} + C_{\varepsilon/\beta}(\beta^{2-\frac{d}{2}}))$. Passing to the limit along the sequence ε_k by using Proposition II.3, we deduce the equality in law

$$\lambda_n(Q_{L/\beta}, \beta^{2-\frac{d}{2}}\xi) = \beta^2 \lambda_n(Q_L, \xi) - \delta_\beta \ ,$$

where $\delta_\beta := \lim_{\varepsilon\downarrow 0} \Big(\beta^2 C_\varepsilon(1) - C_{\varepsilon/\beta}(\beta^{2-\frac{d}{2}})\Big)$. From the asymptotic expressions of the renormalisation constants collected in Appendix II.A, we can deduce that $\delta_\beta = 0$ in dimension 1 while in dimensions 2 and 3, $\delta_\beta = O(\beta^2 \ln \beta^{-1})$ as $\beta \downarrow 0$. ☐

Finally, we state an estimate that allows to approximate, from above and below, the main eigenvalue over Q_L in terms of the main eigenvalues over smaller boxes. This is a general result, which is originally due to Gärtner and König [GK00] in the case where the potential is smooth.

Proposition II.7 (Estimation by division into sub-boxes. See Appendix II.C)**.** *There exists a constant $K > 0$ such that for all $\beta > 0$ and all $L > r \geq 1$, we have almost surely*

$$\min_{k\in\mathbb{Z}^d : |k|_\infty \leq \frac{L}{2r} + \frac{3}{4}} \lambda_1(rk + Q_{\frac{3r}{2}}, \beta\xi) - \frac{K}{r^2} \leq \lambda_1(Q_L, \beta\xi) \leq \min_{k\in\mathbb{Z}^d : |k|_\infty < \frac{L}{2r} - \frac{1}{2}} \lambda_1(rk + Q_r, \beta\xi) \quad \text{(II.7)}$$

II.2.2 A large deviation estimate

The proof of the tail estimates stated in Theorem II.2 revolves around the following large deviations estimate for the main eigenvalue. Its proof requires notions from the theory of regularity structures and is therefore postponed to the next section. From now on, we will often abbreviate $\lambda_1(Q,V)$ into $\lambda(Q,V)$.

Proposition II.8 (See section II.3)**.** *Fix $L \geq 1$. The collection of random variables $(\lambda(Q_L,\beta\xi))_{\beta>0}$ satisfies for all $c \in \mathbb{R}$ the following large deviations estimate:*

$$-\inf_{x\in(-\infty,c)} I_L(x) \leq \liminf_{\beta\to 0} \beta^2 \log \mathbb{P}\Big(\lambda(Q_L,\beta\xi) \in (-\infty,c)\Big) \leq \limsup_{\beta\to 0} \beta^2 \log \mathbb{P}\Big(\lambda(Q_L,\beta\xi) \in (-\infty,c]\Big) \leq -\inf_{x\in(-\infty,c]} I_L(x)\ ,$$

where the rate function $I_L : \mathbb{R} \to [0,\infty]$ is defined by

$$I_L(x) = \inf\left\{\frac{1}{2}\|V\|^2_{L^2(Q_L)} : V \in L^2(Q_L), \lambda(Q_L,V) = x\right\}\ , \quad x \in \mathbb{R}\ .$$

Remark II.9. For any $x \in \mathbb{R}$ and any box Q, there exists V such that $\lambda(Q,V) = x$. Indeed, it suffices to take V constant equal to $x - x_0$ where $x_0 := \lambda(Q,0)$ is the lowest eigenvalue of $-\Delta$ on Q.

We will use the notation $\mathrm{LD}(\beta^2, I_L)$ as a shortcut for the large deviations estimate of rate β^2 and rate function I_L. We adopt the notation $I_L(J) := \inf_{x\in J} I_L(x)$ for any set $J \subset \mathbb{R}$ and we define the constant

$$\rho := \inf_{L>0} I_L((-\infty,-1]) = \lim_{L\to\infty} I_L((-\infty,-1])\ , \tag{II.8}$$

where the second equality comes from the fact that $L \mapsto I_L((-\infty,x])$ is non-increasing.

Proposition II.10 (Study of ρ. See section II.2.3)**.** *The following properties hold:*

1. *For all $L \geq 1$, the map $x \mapsto I_L((-\infty,x])$ is continuous.*
2. *For all $b \in \mathbb{R}$ and all real sequence $(a_L)_{L\geq 1}$ such that $L^{d/2}a_L \to 0$ as $L \to \infty$, we have*
$$\lim_{L\to\infty} I_L((-\infty,b+a_L]) = \lim_{L\to\infty} I_L((-\infty,b]) = \lim_{L\to\infty} I_L((-\infty,b)).$$
3. *The constant ρ can be evaluated through a variational problem :*
$$\rho = \frac{1}{2}\left\{\sup_{\substack{\psi\in H^1(\mathbb{R}^d)\\ \|\psi\|_{L^2}=1}} \Big(\|\psi\|^2_{L^4} - \|\nabla\psi\|^2_{L^2}\Big)\right\}^{-(2-d/2)} \tag{II.9}$$
In particular, we have $C_d = d/\rho = \Big(d^{1+d/2}(4-d)^{2-d/2}/8\Big)\kappa_d^4$. Moreover, optimizers for this variational problem are properly rescaled optimizers of the Gagliardo-Nirenberg inequality.

II.2.3 Properties of the rate function

Recall the Gagliardo-Nirenberg inequality (II.1) and the associated optimal constant κ_d defined in (II.2). In the study of the rate function of the large deviation estimate will appear a variational problem that is closely related to the Gagliardo-Nirenberg inequality: in the following result we collect a few facts on this variational problem.

Lemma II.11. *We have*

$$\sup_{\substack{\psi\in H^1(\mathbb{R}^d)\\ \|\psi\|_{L^2}=1}} \left(\|\psi\|_{L^4}^2 - \|\nabla\psi\|_{L^2}^2\right) = \left(\frac{d}{4}\right)^{\frac{d}{4-d}} \left(\frac{4-d}{4}\right) \kappa_d^{8/(4-d)} = \left(\frac{C_d}{2d}\right)^{\frac{1}{2-d/2}} , \qquad \text{(II.10)}$$

where C_d is given by (II.4).

Moreover, a function w with unit L^2 norm is an optimizer of (II.10) *if and only if $w = \lambda^{d/2}u(\lambda\cdot)$ with $\lambda^{2-d/2} = \frac{d\|u\|_{L^4}^2}{4\|\nabla u\|_{L^2}^2}$, where u is an optimizer of the Gagliardo-Nirenberg inequality with unit L^2 norm.*

Finally, we also have

$$\sup_{\substack{\psi\in H^1(\mathbb{R}^d)\\ \|\psi\|_{L^2}=1}} \left(\|\psi\|_{L^4}^2 - \|\nabla\psi\|_{L^2}^2\right) = \sup_{\substack{V\in L^2(\mathbb{R}^d)\\ \|V\|_{L^2}=1}} -\lambda(\mathbb{R}^d, V) . \qquad \text{(II.11)}$$

Proof. For any $u \in H^1(\mathbb{R}^d)$ such that $\|u\|_{L^2} = 1$, we write $J(u) := \|u\|_{L^4}^2 - \|\nabla u\|_{L^2}^2$. Consider the family of functions $w(x) = \lambda^{d/2}u(\lambda x)$ indexed by $\lambda > 0$. Plugging w into the functional J gives us

$$J(w) = \lambda^{d/2} \|u\|_{L^4}^2 - \lambda^2 \|\nabla u\|_{L^2}^2 .$$

for all $\lambda > 0$. The RHS is maximized for $\lambda^{2-d/2} = \frac{d\|u\|_{L^4}^2}{4\|\nabla u\|_{L^2}^2}$, which results in the identity

$$\sup_{\lambda>0} J(w) = \left(\frac{d}{4}\right)^{\frac{d}{4-d}} \left(\frac{4-d}{4}\right) \left(\frac{\|u\|_{L^4}}{\|\nabla u\|_{L^2}^{d/4}}\right)^{\frac{8}{4-d}} .$$

We distinguish two cases. Either u (and therefore all the w's) optimizes the Gagliardo-Nirenberg inequality (II.1), in which case

$$\sup_{\lambda>0} J(w) = \left(\frac{d}{4}\right)^{\frac{d}{4-d}} \left(\frac{4-d}{4}\right) \kappa_d^{\frac{8}{4-d}} .$$

Or u (and therefore all the w's) are not optimizers of the Gagliardo-Nirenberg inequality, and then

$$\sup_{\lambda>0} J(w) < \left(\frac{d}{4}\right)^{\frac{d}{4-d}} \left(\frac{4-d}{4}\right) \kappa_d^{\frac{8}{4-d}} .$$

This proves (II.10). Moreover, this shows that w is an optimizer of J (over the functions of unit L^2 norm) if and only if $w = \lambda^{d/2}u(\lambda\cdot)$ with $\lambda^{2-d/2} = \frac{d\|u\|_{L^4}^2}{4\|\nabla u\|_{L^2}^2}$ where u is an optimizer of the Gagliardo-Nirenberg inequality with unit L^2 norm.

To prove the final identity of the statement, note that

$$\sup_{\substack{V\in L^2(\mathbb{R}^d)\\ \|V\|_{L^2}=1}} -\lambda(\mathbb{R}^d,V) = \sup_{\substack{\psi\in H^1\\ \|\psi\|_{L^2}=1}} \sup_{\substack{V\in L^2(\mathbb{R}^d)\\ \|V\|_{L^2}=1}} \int \Big(-|\nabla\psi|^2 - V\psi^2\Big) = \sup_{\substack{\psi\in H^1(\mathbb{R}^d)\\ \|\psi\|_{L^2}=1}} \Big(\|\psi\|_{L^4}^2 - \|\nabla\psi\|_{L^2}^2\Big)$$

where we have chosen $V = -\psi^2/\left\|\psi\right\|_{L^4}^2$ to attain the supremum over V. □

Proof of Proposition II.10. 1. & 2. Notice that for $V \in L^2$ and $a \in \mathbb{R}$, we have $\lambda(Q_L, V + a\mathbf{1}_{Q_L}) = \lambda(Q_L,V) + a$ and that for all $\delta > 0$, $\pm 2\langle a\mathbf{1}_{Q_L}, V\rangle = \pm 2\langle (a/\sqrt{\delta})\mathbf{1}_{Q_L}, \sqrt{\delta}V\rangle \leq \delta\left\|V\right\|^2 + \delta^{-1}a^2L^d$. As a result,

$$\begin{aligned} I_L\Big((-\infty, b+a]\Big) &= \inf\left\{\frac{1}{2}\left\|V\right\|^2 : \lambda(Q_L,V) \leq b+a\right\} \\ &= \inf\left\{\frac{1}{2}\left\|V\right\|^2 : \lambda(Q_L, V - a\mathbf{1}_{Q_L}) \leq b\right\} \\ &= \inf\left\{\frac{1}{2}\left\|V + a\mathbf{1}_{Q_L}\right\|^2 : \lambda(Q_L,V) \leq b\right\} \\ &\begin{cases} \leq (1+\delta)I_L((-\infty,b]) + \frac{1}{2}(1+\delta^{-1})a^2L^d \\ \geq (1-\delta)I_L((-\infty,b]) + \frac{1}{2}(1-\delta^{-1})a^2L^d \end{cases} \end{aligned}$$

This easily entails 1. and the first equality of 2. Combining the following inequalities

$$\inf I_L((-\infty,b]) \leq \inf I_L((-\infty,b)) \leq \inf I_L((-\infty, b - L^{-d}]),$$

with the arguments above yields the second equality of 2.

3. Given (II.11), the statement follows if we can establish

$$\rho = \frac{1}{2} \inf_{\substack{V\in L^2(\mathbb{R}^d)\\ \lambda(\mathbb{R}^d,V)\leq -1}} \left\|V\right\|_{L^2}^2 \ , \tag{II.12}$$

and

$$\inf_{\substack{V\in L^2(\mathbb{R}^d)\\ \lambda(\mathbb{R}^d,V)\leq -1}} \left\|V\right\|_{L^2}^2 = \inf_{\substack{V\in L^2(\mathbb{R}^d)\\ \lambda(\mathbb{R}^d,V)= -1}} \left\|V\right\|_{L^2}^2 = \left\{ \sup_{\substack{W\in L^2(\mathbb{R}^d)\\ \|W\|_{L^2}=1}} -\lambda(\mathbb{R}^d,W) \right\}^{-(2-\frac{d}{2})} . \tag{II.13}$$

Both equalities in (II.13) are proven by an argument of scaling. Let us start with the second. By Lemma II.11, the rightmost term of (II.13) is strictly positive and therefore

$$\left\{ \sup_{\substack{W\in L^2(\mathbb{R}^d)\\ \|W\|_{L^2}=1}} -\lambda(\mathbb{R}^d,W) \right\}^{-(2-\frac{d}{2})} = \left\{ \sup_{\substack{W\in L^2(\mathbb{R}^d)\\ \|W\|_{L^2}=1\\ \lambda(\mathbb{R}^d,W)<0}} -\lambda(\mathbb{R}^d,W) \right\}^{-(2-\frac{d}{2})} = \inf_{\substack{W\in L^2(\mathbb{R}^d)\\ \|W\|_{L^2}=1\\ \lambda(\mathbb{R}^d,W)<0}} \frac{1}{[-\lambda(\mathbb{R}^d,W)]^{2-\frac{d}{2}}} \tag{II.14}$$

Consequently, it suffices to show that

$$\inf_{\substack{V\in L^2(\mathbb{R}^d)\\ \lambda(\mathbb{R}^d,V)=-1}} \|V\|_{L^2}^2 = \inf_{\substack{W\in L^2(\mathbb{R}^d)\\ \|W\|_{L^2}=1\\ \lambda(\mathbb{R}^d,W)<0}} \frac{1}{[-\lambda(\mathbb{R}^d,W)]^{2-d/2}}\,. \tag{II.15}$$

Consider the map $V\mapsto W$ defined by: for any $V\in L^2(\mathbb{R}^d)$ such that $\lambda(\mathbb{R}^d,V)=-1$, set $W(x)=r^2V(rx)$ with r defined by $r^{4-d}=\|V\|_{L^2}^{-2}$. A simple computation shows that $-\lambda(\mathbb{R}^d,W)=-r^2\lambda(\mathbb{R}^d,V)=r^2$ and $\|W\|_{L^2}^2=r^{4-d}\|V\|_{L^2}^2=1$. It is thus immediate to deduce that our map $V\mapsto W$ is a bijection between the two sets that appear in (II.15), and by construction, we have $\|V\|_{L^2}^2=\frac{1}{[-\lambda(\mathbb{R}^d,W)]^{2-d/2}}$. The second equality of (II.13) is thus proved.

We turn to the first equality in (II.13). Recall that for $d\le 3$, the operator $-\Delta+V$ is bounded below whenever $V\in L^2$. Consequently $\lambda(\mathbb{R}^d,V)>-\infty$. Take $V\in L^2(\mathbb{R}^d)$ such that $\lambda(\mathbb{R}^d,V)<-1$ and set $W(x)=r^2V(rx)$ with r defined by $r^2=-1/\lambda(\mathbb{R}^d,V)$. The previous computations show that $\lambda(\mathbb{R}^d,W)=-1$ and $\|V\|_{L^2}^2>\|W\|_{L^2}^2$. Hence,

$$\inf_{\substack{V\in L^2(\mathbb{R}^d)\\ \lambda(\mathbb{R}^d,V)\le-1}} \|V\|_{L^2}^2 \ge \inf_{\substack{W\in L^2(\mathbb{R}^d)\\ \lambda(\mathbb{R}^d,W)=-1}} \|W\|_{L^2}^2\,.$$

The converse inequality is obvious.

Now it remains to show (II.12), or in other words,

$$\inf_{\substack{V\in L^2(\mathbb{R}^d)\\ \lambda(\mathbb{R}^d,V)\le-1}} \|V\|_{L^2}^2 = \inf_{L>0}\inf_{\substack{V\in L^2(Q_L)\\ \lambda(Q_L,V)\le-1}} \|V\|_{L^2(Q_L)}^2\,.$$

The fact that l.h.s. $\le$ r.h.s. is a direct consequence of the inequality $\lambda(\mathbb{R}^d,V\mathbf{1}_{Q_L})\le\lambda(Q_L,V)$. On the other hand, fix any $\varepsilon>0$ and pick $V_\varepsilon\in L^2(\mathbb{R}^d)$ such that $\lambda(\mathbb{R}^d,V_\varepsilon)\le -1$ and

$$\|V_\varepsilon\|_{L^2}^2 < (1+\varepsilon)\inf_{\substack{V\in L^2(\mathbb{R}^d)\\ \lambda(\mathbb{R}^d,V)\le-1}} \|V\|_{L^2}^2\;.$$

Let $\bar V_\varepsilon := \alpha^2 V_\varepsilon(\alpha\cdot)$ with $\alpha=\sqrt{1+\varepsilon}$. By the scaling property $\lambda(\mathbb{R}^d,\bar V_\varepsilon)=\alpha^2\lambda(\mathbb{R}^d,V_\varepsilon)$, we have

$$\begin{aligned}\left\|\bar V_\varepsilon\right\|_{L^2}^2 &= \alpha^{4-d}\|V_\varepsilon\|_{L^2}^2 < (1+\varepsilon)^{3-\frac{d}{2}} \inf_{\substack{V\in L^2(\mathbb{R}^d)\\ \lambda(\mathbb{R}^d,V)\le-1}} \|V\|_{L^2}^2\\ \lambda(\mathbb{R}^d,\bar V_\varepsilon) &= \alpha^2\lambda(\mathbb{R}^d,V_\varepsilon)\le-\alpha^2=-1-\varepsilon\end{aligned}$$

It can be checked that as $L\to\infty$, $\lambda(Q_L,\bar V_\varepsilon)\to\lambda(\mathbb{R}^d,\bar V_\varepsilon)$, and therefore for all L large enough we have

$$\lambda(Q_L,\bar V_\varepsilon)\le -1.$$

thus implying for all L large enough

$$\inf_{\substack{V\in L^2(Q_L)\\ \lambda(Q_L,V)\le-1}} \|V\|_{L^2}^2 \le \left\|\bar V_\varepsilon\right\|_{L^2(Q_L)}^2 \le \left\|\bar V_\varepsilon\right\|_{L^2(\mathbb{R}^d)}^2 \le (1+\varepsilon)^{3-\frac{d}{2}} \inf_{\substack{V\in L^2(\mathbb{R}^d)\\ \lambda(\mathbb{R}^d,V)\le-1}} \|V\|_{L^2}^2\,.$$

(II.12) is then proved by taking an infimum over L and making ε shrink to zero.

Combining (II.12), (II.11) and (II.13), (II.9) is then established, whereas the expression for C_d and the properties satisfied by the optimizer are consequences of Lemma II.11. □

II.2.4 Proofs of theorems II.1 and II.2

We can now proceed to the proof of the tail estimates.

Proof of Theorem II.2. We will only consider $n = 1$ as this is the only case needed for the proof of Theorem II.1. To treat the general case $n \geq 1$, it suffices to argue as in the proof of Theorem II.1 below, see in particular (II.17). For $x > 0$, we write $\beta = 1/\sqrt{x}$. We begin by applying the scaling property of Proposition II.5:

$$\mathbb{P}\left(\lambda_1(Q_L, \xi) \geq -x\right) = \mathbb{P}\left(\beta^2 \lambda_1(Q_L, \xi) \geq -1\right) = \mathbb{P}\left(\lambda_1(Q_{L/\beta}, \beta^{2-\frac{d}{2}}\xi) + \delta_\beta \geq -1\right).$$

At this point, for any $1 \leq r \leq L/\beta$ we squeeze the domain $Q_{L/\beta}$ in between unions of boxes of size r and $\frac{3r}{2}$ respectively:

$$\bigcup_{k \in \mathbb{Z}^d : |k|_\infty \leq \frac{L}{2\beta r} - \frac{1}{2}} \{rk + Q_r\} \quad \subset \quad Q_{L/\beta} \quad \subset \quad \bigcup_{k \in \mathbb{Z}^d : |k|_\infty \leq \frac{L}{2\beta r} + \frac{3}{4}} \{rk + Q_{\frac{3r}{2}}\}\,.$$

Note that the number of boxes in both unions is of order $(\frac{L}{\beta r})^d$ as $\beta \downarrow 0$ uniformly over all $L \geq 1$, so that for any given $c_1 < 1 < c_2$ there exists $\beta_1 = \beta_1(r)$ such that for all $\beta < \beta_1$ and all $L \geq 1$ these two numbers are comprised in between $c_1(\frac{L}{\beta r})^d$ and $c_2(\frac{L}{\beta r})^d$. Applying the estimation by division into small boxes of Proposition II.7 at the first line and the independence property of Proposition II.5 at the second line, we thus get for all $x \geq x_1 := \beta_1^{-2}$

$$\begin{aligned}\mathbb{P}(\lambda_1(Q_L, \xi) \geq -x) &\leq \mathbb{P}\left(\min_{k \in \mathbb{Z}^d : |k|_\infty < \frac{L}{2\beta r} - \frac{1}{2}} \lambda_1(rk + Q_r, \beta^{2-\frac{d}{2}}\xi) \geq -1 - \delta_\beta\right) \\ &\leq \left[1 - \mathbb{P}\left(\lambda_1(Q_r, \beta^{2-\frac{d}{2}}\xi) < -1 - \delta_\beta\right)\right]^{c_1\left(\frac{L}{\beta r}\right)^d}\,.\end{aligned}$$

Regarding the lower bound we also apply Proposition II.7. However the r.v. that appear are no longer independent so we use a union bound at the third line to get

$$\begin{aligned}\mathbb{P}(\lambda_1(Q_L, \xi) \geq -x) &\geq \mathbb{P}\left(\min_{k \in \mathbb{Z}^d : |k|_\infty < \frac{L}{2\beta r} + \frac{3}{4}} \lambda_1(rk + Q_{\frac{3r}{2}}, \beta^{2-\frac{d}{2}}\xi) \geq -1 - \delta_\beta + \frac{K}{r^2}\right) \\ &\geq 1 - \mathbb{P}\left(\min_{k \in \mathbb{Z}^d : |k|_\infty < \frac{L}{2\beta r} + \frac{3}{4}} \lambda_1(rk + Q_{\frac{3r}{2}}, \beta^{2-\frac{d}{2}}\xi) < -1 - \delta_\beta + \frac{K}{r^2}\right) \\ &\geq 1 - c_2\left(\frac{L}{\beta r}\right)^d \mathbb{P}\left(\lambda_1(Q_{\frac{3r}{2}}, \beta^{2-\frac{d}{2}}\xi) < -1 - \delta_\beta + \frac{K}{r^2}\right)\,.\end{aligned}$$

Since δ_β goes to 0 deterministically as $\beta \to 0$ and given the continuity of the rate function stated as item 1. of Proposition II.10, the term δ_β does not affect large deviation behaviors. Consequently the large deviations estimate of Proposition II.8 implies that for fixed r and as $\beta \downarrow 0$

$$\lambda_1(Q_r, \beta^{2-\frac{d}{2}}\xi) + \delta_\beta \sim \mathrm{LD}(\beta^{4-d}, I_r), \quad \lambda_1(Q_{\frac{3r}{2}}, \beta^{2-\frac{d}{2}}\xi) + \delta_\beta \sim \mathrm{LD}(\beta^{4-d}, I_{\frac{3r}{2}}).$$

Moreover the properties on the rate function collected in Proposition II.10 show that

$$\rho = \lim_{r\to\infty} I_r\Big((-\infty,-1)\Big) = \lim_{r\to\infty} I_{\frac{3r}{2}}\Big((-\infty,-1+K/r^2]\Big) > 0 \,.$$

Fix some $\eta > 0$. Choosing r large enough we thus have

$$\rho(1-\eta) < I_{\frac{3r}{2}}\Big((-\infty,-1+K/r^2]\Big)\,, \quad I_r\Big((-\infty,-1)\Big) < \rho(1+\eta)\,.$$

The aforementioned large deviations estimates imply that for all r large enough, we have

$$\limsup_{\beta\downarrow 0} \beta^{4-d}\log\mathbb{P}\left(\lambda_1(Q_{\frac{3r}{2}}, \beta^{2-\frac{d}{2}}\xi) < -1-\delta_\beta+\frac{K}{r^2}\right) \le -I_{\frac{3r}{2}}\left(\left(-\infty,-1+\frac{K}{r^2}\right]\right) < -\rho(1-\eta),$$
$$\liminf_{\beta\downarrow 0} \beta^{4-d}\log\mathbb{P}\left(\lambda_1(Q_r, \beta^{2-\frac{d}{2}}\xi) < -1-\delta_\beta\right) \ge -I_r((-\infty,-1)) > -\rho(1+\eta)\,.$$

We can thus find $\beta_0 \le \beta_1$ such that for all $\beta \le \beta_0$ (that is, for all $x \ge x_0 := \beta_0^{-2}$):

$$1 - c_2\left(\frac{L}{\beta r}\right)^d e^{-(1-\eta)\rho\beta^{-4+d}} \le \mathbb{P}\Big(\lambda_1(Q_L,\xi) \ge -x\Big) \le \Big(1-e^{-(1+\eta)\rho\beta^{-4+d}}\Big)^{c_1(\frac{L}{\beta r})^d}\,.$$

Since we have $e^{-2y} \le 1-y \le e^{-y}$ for all $y \ge 0$ small enough, the previous inequalities yield (up to possibly diminishing β_0):

$$\exp\left(-\frac{2c_2}{\beta^d r^d}e^{d\log L-(1-\eta)\rho\beta^{-4+d}}\right) \le \mathbb{P}(\lambda_1(Q_L,\xi)\ge -x) \le \exp\left(-\frac{c_1}{\beta^d r^d}e^{d\log L-(1+\eta)\rho\beta^{-4+d}}\right).$$

Setting $\gamma_1 = c_1/r^d$, $\gamma_2 = 2c_2/r^d$, and replacing β by $1/\sqrt{x}$, we obtain the desired bound for $n=1$. □

With Theorem II.2 at hand, we are now able to prove our main result.

Proof of Theorem II.1. In this proof, we set for convenience the quantity $a_L := (C_d \log L)^{\frac{1}{2-\frac{d}{2}}}$ for $L \ge 1$ (which is the first term in (II.6)). Assume that for all $\delta > 0$ and all $n \ge 1$

$$\mathbb{P}\left(\liminf_{m\to\infty}\{-(1+\delta)\,a_{2^m} \le \lambda_{n,2^m} \le -(1-\delta)\,a_{2^m}\}\right) = 1\,. \tag{II.16}$$

To deduce the statement of the theorem, it suffices to extend this asymptotic from $L \in \{2^m, m \ge 1\}$ to general $L \in [1,\infty)$. This can be done as follows. For all $L \ge 1$, there exists $m \in \mathbb{N}$ such that $2^m \leq L < 2^{m+1}$. By item 2. of Proposition II.3, on the event $\Omega_0(1)$ and provided $\lambda_{n,2^m} \leq 0$ we have

$$\frac{\lambda_{n,2^{m+1}}}{a_{2^m}} \leq \frac{\lambda_{n,L}}{a_L} \leq \frac{\lambda_{n,2^m}}{a_{2^{m+1}}} .$$

Since $a_{2^m}/a_{2^{m+1}} \to 1$ as $m \to \infty$, we deduce from (II.16) that the middle term goes to -1 almost surely as $L \to \infty$.
We are left with the proof of (II.16). By item 3. of Proposition II.3 we have on the event $\Omega_0(1)$ and for all $n \geq 1$

$$\lambda_{1,L} \leq \lambda_{n,L} \leq \max_{1 \leq i \leq n} \lambda_{(i)} \tag{II.17}$$

where $\lambda_{(i)}$ is the principal eigenvalue of the operator $\mathcal{H}(Q_{(i)}, \xi)$ and the boxes $Q_{(i)}$ are n disjoint sub-boxes of Q_L whose side-lengths are L/n. By item 3. of Proposition II.5, the $\lambda_{(i)}$'s are i.i.d. with the same law as $\lambda_{1,L/n}$. Specialising these inequalities to $L = 2^m$, we deduce that (II.16) follows from

$$\mathbb{P}\left(\limsup_{m\to\infty} \{\lambda_{1,2^m} < -(1+\delta)\, a_{2^m}\}\right) = 0 \,, \tag{II.18}$$

$$\mathbb{P}\left(\limsup_{m\to\infty} \left\{\max_{1\leq i\leq n} \lambda_{(i)} > -(1-\delta)\, a_{2^m}\right\}\right) = 0 \,. \tag{II.19}$$

Take $\eta > 0$ such that $(1-\eta)(1+\delta)^{2-\frac{d}{2}} > 1$. By Theorem II.2 and using the inequality $1 - e^{-y} \leq y$ that holds for all $y \in \mathbb{R}$, we find for all $m \geq 1$ large enough

$$\begin{aligned}\mathbb{P}\left(\lambda_{1,2^m} < -(1+\delta)\, a_{2^m}\right) &= 1 - \mathbb{P}\left(\lambda_{1,2^m} \geq -(1+\delta)\, a_{2^m}\right) \\ &\leq \gamma_2 (1+\delta)^{\frac{d}{2}} \left(C_d \log 2^m\right)^{\frac{d/2}{2-\frac{d}{2}}} 2^{-md[(1-\eta)(1+\delta)^{2-\frac{d}{2}}-1]} .\end{aligned}$$

The Borel-Cantelli Lemma allows to deduce (II.18).
Recall that the $\lambda_{(i)}$'s are i.i.d. with the same law as $\lambda_{1,2^m/n}$ so that

$$\mathbb{P}\left(\max_{1\leq i\leq n} \lambda_{(i)} > -(1-\delta)\, a_{2^m}\right) \leq n\, \mathbb{P}\left(\lambda_{1,2^m/n} > -(1-\delta)\, a_{2^m}\right) .$$

Take $\eta > 0$ such that $(1+\eta)(1-\delta)^{2-\frac{d}{2}} < 1$. By Theorem II.2 again, we thus find for all $m \geq 1$ large enough

$$\mathbb{P}\left(\lambda_{1,2^m/n} > -(1-\delta)\, a_{2^m}\right) \leq \exp\left[-\gamma_1 n^{-d} (1-\delta)^{\frac{d}{2}} \left(C_d \log 2^m\right)^{\frac{d/2}{2-\frac{d}{2}}} 2^{md[1-(1+\eta)(1-\delta)^{2-\frac{d}{2}}]}\right] .$$

Applying again the Borel-Cantelli Lemma, we deduce (II.19), thus concluding the proof of Theorem II.1. □

II.2.5 About conjecture 2

Recall that the optimizers of the Gagliardo-Nirenberg inequality are exactly the functions $aQ(bx+c)$ with $a, b \in \mathbb{R}\backslash\{0\}, c \in \mathbb{R}^d$ where Q is the unique positive radial solution of $-\Delta Q - Q^3 = -Q$.

Proposition II.12. *The optimizers of* (II.10) *coincide with the set of functions* $\{\pm w_*(\cdot + c), c \in \mathbb{R}^d\}$, *where*

$$w_* := \frac{\mu^{d/2}}{\|Q\|_{L^2}} Q(\mu\,\cdot), \quad \mu = \left(\frac{C_d}{2d}\right)^{\frac{1}{4-d}} .$$

Proof. From Lemma II.11, we know that any optimizer w is of the form $w = \lambda^{d/2}u(\lambda\cdot)$ where $\lambda^{2-d/2} = \frac{d\|u\|_{L^4}^2}{4\|\nabla u\|_{L^2}^2}$ and u is a Gagliardo-Nirenberge optimizer with unit L^2-norm. Also, from the property recalled at the beginning of this subsection, any such u takes the form $aQ(bx+c)$ with $a = \pm|b|^{d/2}\|Q\|_{L^2}^{-1}$, $b \in \mathbb{R}\backslash\{0\}$ and $c \in \mathbb{R}^d$. A computation then shows that w is an optimizer to (II.10) if and only if it is of the form $\pm\frac{|b\lambda|^{d/2}}{\|Q\|_{L^2}}Q(b\lambda\cdot+c)$ where $|b\lambda|$ is a fixed value given by $\left(\frac{d\|Q\|_{L^4}^2}{4\|\nabla Q\|_{L^2}^2}\right)^{\frac{2}{2-d/2}}$. Set $w_* = \frac{|b\lambda|^{d/2}}{\|Q\|_{L^2}}Q(|b\lambda|\cdot)$. Then indeed all optimizers to (II.10) are given by $\{\pm w_*(\cdot+c), c \in \mathbb{R}^d\}$.

The value of $|b\lambda|$ can be determined by an indirect argument. Recall that (II.10) can be reformulated into the constrained variational problem $\tilde{J}(w,t) = J(w) - t(\|w\|_{L^2}^2 - 1)$, where J is the functional introduced in the proof of Lemma II.11 and $t \in \mathbb{R}$ is the Lagrange multiplier. Routine variational calculus arguments (taking $\delta\tilde{J} = 0$, see for example [Amb92, page 9]) show that any optimizer w satisfies the PDE

$$\begin{cases} -\Delta w - \frac{w^3}{\|w\|_{L^4}^2} = -\left(\frac{C_d}{2d}\right)^{\frac{1}{2-d/2}} w \\ \|w\|_{L^2} = 1 \end{cases}$$

of which w_* is a positive, radial solution. Set $v(x) = (\mu\|w_*\|_{L^4})^{-1}w(x/\mu)$ with μ as in the statement. Recall from the introduction that $\|Q\|_{L^4}^4 = \frac{2d}{C_d}$. A computation shows that v is a positive, radial solution of

$$\begin{cases} -\Delta v - v^3 = -v \\ \|v\|_{L^4}^4 = \mu^{d-4} = \frac{2d}{C_d} \end{cases},$$

and therefore $v = Q$. This leads to the identity

$$v(x) = \left(\frac{|b\lambda|}{\mu}\right)^{d/4} Q(|b\lambda|x) = Q(x) ,$$

which evaluated at $x = 0$, and given $Q(0) \neq 0$ since Q is positive, ensures that $|b\lambda| = \mu$ and therefore

$$w_*(x) = \frac{|b\lambda|^{d/2}}{\|Q\|_{L^2}}Q(|b\lambda|x) = \frac{\mu^{d/2}}{\|Q\|_{L^2}}Q(\mu x) .$$

□

Take $\beta^2 = 1/a_L$. By the rescaling property stated in Proposition II.5, the event $\{\lambda(Q_L,\xi) \asymp -\beta^{-2}\}$ is "equivalent" to the event $\{\lambda(Q_{L/\beta}, \beta^2\xi(\cdot\beta)) \asymp -1\}$. (Recall that $\beta^2\xi(\cdot\beta)$ has the same law as $\beta^{2-\frac{d}{2}}\xi$.) By the subdivision into sub-boxes of Proposition II.7, the latter event "coincides" with

$$\left\{ \min_{|k|\leq \frac{L}{2\beta r}} \lambda\Big(kr + Q_r, \beta^2\xi(\cdot\beta)\Big) \asymp -1 \right\} .$$

The r.v. involved in the min are independent. For such an event to be satisfied, typically only one of these r.v. is of order -1. The large deviation estimate of Proposition II.8 shows that the probability of $\{\lambda(Q_r, \beta^2\xi(\cdot\beta)) \asymp -1\}$ is roughly

$$\exp(-\beta^{d-4} I_r(-\infty,-1)) .$$

Heuristically, to achieve $\{\lambda(Q_r, \beta^2\xi(\cdot\beta)) \asymp -1\}$, one needs $\beta^2\xi(\cdot\beta) \asymp V_*$ where V_* is the argmin of I_r. Taking r large enough, this argmin should be close to the argmin of I_∞ and we can thus assume that V_* is the argmin of I_∞.
The computations in the proofs of Proposition II.10 and Lemma II.11 show that $V_*(y) = \mu^{-2}W_*(y/\mu)$, with $\mu = \left(\frac{C_d}{2d}\right)^{\frac{1}{4-d}}$ and $W_*(u) = -\frac{w_*^2(u)}{\|w_*\|^2_{L^4}}$ where w_* is the radial optimizer of (II.10) with unit L^2 norm defined in Proposition II.12. From Proposition II.12, we deduce $W_* = -\mu^2 Q^2(\mu\cdot)$ and that $V_* = -Q^2 = -\frac{{\psi_*}^2}{\|\psi_*\|^2_{L^4(\mathbb{R}^d)}}\sqrt{\frac{2d}{C_d}}$ where $\psi_* = Q/\|Q\|_{L^2}$. Note that $\lambda(\mathbb{R}^d, V_*) = -1$ and that

$$-\Delta\psi_* + V_*\psi_* = -\psi_* ,$$

so that ψ_* is the eigenfunction associated to the smallest eigenvalue of $-\Delta + V_*$.
The above discussion suggests that ξ has a deterministic behavior at space-scale $1/\sqrt{a_L}$ around $U_{n,L}$:

$$\left(\frac{1}{a_L}\xi\Big(U_{n,L} + \frac{x}{\sqrt{a_L}}\Big), x\in\mathbb{R}^d\right) \Rightarrow V_* .$$

If we assume that for some function ψ (of unit L^2 norm) we have

$$\left(\frac{1}{a_L^{d/4}}|\varphi_{n,L}|\Big(U_{n,L} + \frac{x}{\sqrt{a_L}}\Big), x\in\mathbb{R}^d\right) \Rightarrow \psi ,$$

then a formal passage to the limit as $L\to\infty$ on

$$-\Delta\varphi_{n,L} + \xi\varphi_{n,L} = \lambda_{n,L}\varphi_{n,L} ,$$

yields

$$-\Delta\psi + V_*\psi = -\psi ,$$

and we deduce that $\psi = \psi_*$.

II.3 REGULARITY STRUCTURES AND THE LARGE DEVIATIONS ESTIMATE

Let us briefly summarize the construction of $\mathcal{H} = \mathcal{H}(Q, \beta\xi)$ carried out in [Lab19]. It consists in constructing the resolvents $(\mathcal{H} - a)^{-1}$ associated to this operator. The advantage of dealing with the resolvents is that they satisfy, at least formally, an SPDE that one can hope to solve. Indeed, formal computations show that for any $g \in L^2(Q)$, $(\mathcal{H} + a)^{-1}g$ should be a fixed point of the map

$$u \mapsto (-\Delta + a)^{-1}g - (-\Delta + a)^{-1}(\beta\xi u) \; , \tag{II.20}$$

where $(-\Delta + a)^{-1}$ is the resolvent of the Laplacian endowed with Dirichlet b.c. on Q. While the above procedure can be performed by standard arguments when ξ is smooth, the case of white noise is singular. Indeed, an iteration of the fixed point map yields a term with the same regularity as $\xi \cdot (-\Delta + a)^{-1}\xi$, and it happens that such a quantity blows up in dimension $d \geq 2$.

This is where the theory of regularity structures [Hai14] is applied. The first ingredient is the notion of *model*: this is a random object that encapsulates the values of the noise but also of non-linear functionals associated to it. For smooth driving noise ξ, there is an associated *canonical model* that can be constructed generically. On the other hand, for singular ξ such as white noise, some non-linear functionals are ill-defined and the construction of the model is performed through a limiting procedure: starting from a regularised noise ξ_ε for which all non-linear functionals are well-defined, one subtracts some renormalisation constants that typically diverge as $\varepsilon \downarrow 0$, but that allow to pass to the limit on the non-linear functionals.
The second ingredient is a calculus developed at the level of the so-called modelled distributions, that allows to prove existence and uniqueness of fixed point maps such as (II.20): this part of the theory will not be needed in the present work.

II.3.1 REGULARITY STRUCTURES

Definition II.13. A regularity structure is a triplet $(\mathcal{A}, \mathcal{T}, \mathcal{G})$ with the following properties:

1. $\mathcal{A} \subset \mathbb{R}$ is a locally finite set of indices that is bounded from below and contains 0.

2. $\mathcal{T} = \bigoplus_{\alpha \in \mathcal{A}} \mathcal{T}_\alpha$ is a graded vector space, where for each $\alpha \in \mathcal{A}$, $\mathcal{T}_\alpha$ is a finite-dimensional Banach space equipped with the norm $\|\cdot\|_\alpha$. In particular, we demand $\mathcal{T}_0 \simeq \mathbb{R}$ with the unit vector $\mathbf{1}$. For any vector τ in a finite-dimensional subspace of $\mathcal{T}$, we write $\|\tau\|$ to be its Euclidean norm.

3. $\mathcal{G}$ is a group acting on $\mathcal{T}$ such that every element Γ of $\mathcal{G}$ satisfies $\Gamma\mathbf{1} = \mathbf{1}$ and for all $\tau \in \mathcal{T}_\alpha$, $\Gamma\tau - \tau \in \mathcal{T}_{<\alpha} := \bigoplus_{\beta \in \mathcal{A}_{<\alpha}} \mathcal{T}_\beta$ where $\mathcal{A}_{<\alpha} = \mathcal{A} \cap (-\infty, \alpha)$.

The prototype for Definition II.13 is the polynomial regularity structure, where $\mathcal{A} = \mathbb{N}$, $\mathcal{T}_n$ is the vector space spanned by all X^k such that $|k| = n$ and $\mathcal{G}$ is the group formed by all the transformations Γ_h that translate any polynomial P by the vector h: $\Gamma_h P(X) = P(X + h)$. (Note that we have used the multi-index notation $X^k = X_1^{k_1} \dots X_d^{k_d}$ and $|k| = k_1 + \cdots + k_d$.)

Let us now introduce the regularity structure associated to the Anderson Hamiltonian, which is an "enlargement" of the polynomial regularity structure. We consider the abstract symbol Ξ that represents ξ at the level of the model space $\mathcal{T}$. We define the sets $\mathcal{F}$ and $\mathcal{U}$, as the smallest sets of symbols such that $\mathcal{F}$ contains the noise symbol Ξ, $\mathcal{U}$ contains all polynomials X^k and

$$\tau \in \mathcal{U} \implies \tau\Xi \in \mathcal{F} \quad \text{and} \quad \tau \in \mathcal{F} \implies \mathcal{I}(\tau) \in \mathcal{U}.$$

Here $\tau\Xi$ and $\mathcal{I}(\tau)$ denote new symbols that need to be considered. In some sense, $\mathcal{U}$ allows to describe the solution u of the fixed point problem while $\mathcal{F}$ allows to describe the product $u\xi$.

Each expression τ is assigned a number $|\tau|$ called homogeneity, which is calculated by the following rules: (1) $|X^k| = |k|$, (2) $|\Xi| = -d/2 - \kappa$ for some fixed $\kappa \in (0, 1/8)$, (3) for any τ, τ', $|\tau\tau'| = |\tau| + |\tau'|$, (4) for any τ, $|\mathcal{I}(\tau)| = |\tau| + 2$. Note that $\mathcal{I}$ stands for an integration map associated to our convolution kernel (the Green function of $-\Delta + m$ for some $m > 0$, see below), which improves regularity by 2 and this explains (4).
Let therefore $\mathcal{A} := \{|\tau| : \tau \in \mathcal{F} \cup \mathcal{U}\}$ and for $\alpha \in \mathcal{A}$, and let $\mathcal{T}_\alpha$ be the vector space spanned by all $\tau \in \mathcal{F} \cup \mathcal{U}$ such that $|\tau| = \alpha$. For the construction of the structure group $\mathcal{G}$, we refer the reader to [Hai14, Sec. 8.1].
From now on, we will *always* restrict $\mathcal{U}$ (resp. $\mathcal{F}$) to symbols whose homogeneities are below some $\gamma \in (3/2, 2 - 4\kappa)$ (resp. $\gamma - d/2 - \kappa$). This yields (in the order of increasing homogeneity):

$$\mathcal{U} = \begin{cases} \{\mathbf{1}, X_i, \mathcal{I}(\Xi)\} & \text{if } d = 1 \\ \{\mathbf{1}, \mathcal{I}(\Xi), X_i\} & \text{if } d = 2 \\ \{\mathbf{1}, \mathcal{I}(\Xi), \mathcal{I}(\Xi\mathcal{I}(\Xi)), X_i, \mathcal{I}(\Xi\mathcal{I}(\Xi\mathcal{I}(\Xi))), \mathcal{I}(\Xi X_i)\} & \text{if } d = 3 \end{cases}$$

On the other hand, the collection $\mathcal{F}$ is obtained by multiplying the elements in $\mathcal{U}$ by Ξ. The regularity structure $(\mathcal{A}, \mathcal{T}, \mathcal{G})$ is now fixed once and for all.

We can define the notion of *admissible models* that, roughly speaking, map abstract objects in $\mathcal{T}$ to genuine distributions. Below we denote by $\mathscr{B}^r$ the space of all $\varphi \in C^r$ supported in the unit ball of $\mathbb{R}^d$ and such that $\|\varphi\|_{C^r} \leq 1$, for some $r > 0$. We also introduce the notation $\varphi_x^\lambda = \lambda^{-d}\varphi((\cdot - x)/\lambda)$. For every $m > 0$, let $P^{(m)}$ be the Green's function of $-\Delta + m$ on $\mathbb{R}^d$, see [Lab19, Sec. 3] for its explicit expression. Following [Lab19, Def 2.1], we consider a decomposition of the kernel $P^{(m)}$ into the sum $P_-^{(m)} + P_+^{(m)}$ where $P_-^{(m)}$ is a smooth function that coincides with $P^{(m)}$ outside $B(0, 2^{-n_m+1})$ and $P_+^{(m)}$ is compactly supported in $B(0, 2^{-n_m+1})$ and coincides with $P^{(m)}$

in $B(0, 2^{-n_m-1})$. Here n_m is the smallest integer such that $2^{-n_m} \leq 1/\sqrt{m}$. We refer to Lemma II.18 for more details.

Definition II.14. Fix $\gamma \in (3/2, 2-4\kappa)$, $r > -\min \mathcal{A} = d/2 + \kappa$ and $m > 0$. An admissible model associated to $P_+^{(m)}$ is a couple $Z = (\Pi, \Gamma)$, where $\Pi = (\Pi_x)_{x \in \mathbb{R}^d}$ is a collection of maps from $\mathcal{T}$ to the space of distributions $\mathscr{D}'(\mathbb{R}^d)$, $\Gamma = (\Gamma_{x,y})_{x,y \in \mathbb{R}^d}$ is a collection of elements of $\mathcal{G}$, that satisfy the following conditions:

- $\Gamma_{x,z} = \Gamma_{x,y}\Gamma_{y,z}$ and $\Pi_x = \Gamma_{x,y}\Pi_y$ for $x, y, z \in \mathbb{R}^d$.
- We have for every $k \in \mathbb{N}^d$, $\Pi_x X^k(y) = (y-x)^k$ together with

$$\Pi_x \mathcal{I}(\tau)(y) = (P_+^{(m)} * \Pi_x \tau)(y) - \sum_{k \in \mathbb{N}^d : |k| < |\mathcal{I}(\tau)|} \partial^k (P_+^{(m)} * \Pi_x \tau)(x) \frac{(y-x)^k}{k!} .$$

- For any given compact set $Q \subset \mathbb{R}^d$, we have

$$\left\| \Pi^{(m)} \right\|_Q := \sup_{x \in Q} \left\| \Pi^{(m)} \right\|_x < \infty ,$$

  where

$$\left\| \Pi^{(m)} \right\|_x := \sup_{\varphi \in \mathscr{B}^r} \sup_{\lambda \in (0, 2^{-n_m}]} \sup_{\alpha \in \mathcal{A}_{<\gamma}} \sup_{\tau \in \mathcal{T}_\alpha} \frac{\left| \left\langle \Pi_x^{(m)} \tau, \varphi_x^\lambda \right\rangle \right|}{\left\| \tau \right\|_\alpha \lambda^\alpha} .$$

- For any given compact set $Q \subset \mathbb{R}^d$, we have

$$\left\| \Gamma^{(m)} \right\|_Q := \sup_{x,y \in Q : |x-y| \leq 2^{-n_m}} \left\| \Gamma^{(m)} \right\|_{x,y} < \infty ,$$

  where

$$\left\| \Gamma^{(m)} \right\|_{x,y} := \sup_{\alpha \leq \beta \in \mathcal{A}_{<\gamma}} \sup_{\tau \in \mathcal{T}_\beta} \frac{\left\| \Gamma_{x,y} \tau \right\|_\alpha}{\left\| \tau \right\|_\beta |x-y|^{\beta - \alpha}} .$$

For any box $Q \subset \mathbb{R}^d$, we set

$$\left\| Z^{(m)} \right\|_Q := \left\| \Pi^{(m)} \right\|_Q + \left\| \Gamma^{(m)} \right\|_Q .$$

Denote by $\mathscr{M}_m$ the space of all admissible models with respect to $P_+^{(m)}$. We equip the space $\mathscr{M}_m$ with the pseudometric

$$\left\| Z; \bar{Z} \right\|_Q := \left\| \Pi - \bar{\Pi} \right\|_Q + \left\| \Gamma - \bar{\Gamma} \right\|_Q \tag{II.21}$$

for $Z = (\Pi, \Gamma)$, $\bar{Z} = (\bar{\Pi}, \bar{\Gamma})$ elements of $\mathscr{M}_m$ and for any given box Q. We can thus consider the quotient space $\mathscr{M}_{m,Q}$ associated to this pseudometric.

For any $V \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$, there exists a unique admissible model $Z^{(m)}(V) = (\Pi^V, \Gamma^V)$ that satisfies $\Pi^V_x \Xi(y) = V(y)$ and for all $\tau \in \mathcal{U}$, $\Pi^V_x \tau\Xi = (\Pi^V_x \tau)(\Pi^V_x \Xi)$, see Appendix II.B. In particular, V can be taken equal to the regularised noise $\beta\xi_\varepsilon := \beta\xi * \rho_\varepsilon$. To alleviate the notations, we will omit the superscript V of Π and Γ, and only express the dependency on the potential function through the notation $Z^{(m)}(V)$ if necessary.
Unfortunately, in dimensions 2 and 3 the corresponding model does not converge as $\varepsilon \downarrow 0$ to an admissible model. However, it was proved in [Lab19] that one can build a *renormalised model* $Z^{(m)}_\varepsilon(\beta\xi_\varepsilon)$ associated to $\beta\xi_\varepsilon$ that converges in probability to an admissible model that we denote $Z^{(m)}(\beta\xi)$. This last model can be interpreted as the model associated with the white noise $\beta\xi$. We refer to Appendix II.A for the definitions of the renormalisation constants and some details on the renormalisation procedure.
In fine, for every $m > 0$, for every box Q and every $\delta > 0$, we have

$$\lim_{\varepsilon\to 0} \mathbb{P}\left(\left\| Z^{(m)}_\varepsilon(\beta\xi_\varepsilon); Z^{(m)}(\beta\xi) \right\|_Q > \delta \right) = 0. \tag{II.22}$$

For convenience we sometimes only write the shorthands $Z^{(m)}$ and $Z^{(m)}_\varepsilon$ without specifying the potential.

Now fix $\beta\xi$ as potential. For later use, we introduce a sequence $(\varepsilon_k)_{k\geq 1}$ and an event Ω' of probability one on which we have convergence *simultaneously* on all boxes Q of the renormalised models $Z^{(m)}_{\varepsilon_k}$ towards $Z^{(m)}$. It follows from a diagonal argument. Recall that $Q_L = (-L/2, L/2)^d$. We construct the sequence $(\varepsilon_k)_{k\geq 1}$ as follows:

1. For $n = 1$, we pick a sequence $(\varepsilon_{1,k})_{k\geq 1}$ such that $\mathbb{P}\left(\left\| Z^{(m)}_{\varepsilon_{1,k}}; Z^{(m)} \right\|_{Q_1} \to 0 \right) = 1$.
2. Having chosen the sequence $(\varepsilon_{n,k})_{k\geq 1}$ such that $\mathbb{P}\left(\bigcap_{i=1}^{n} \left\{ \left\| Z^{(m)}_{\varepsilon_{i,k}}; Z^{(m)} \right\|_{Q_i} \to 0 \right\} \right) = 1$,

   we choose a subsequence $(\varepsilon_{n+1,k})_{k\geq 1}$ of $(\varepsilon_{n,k})_{k\geq 1}$ such that $\mathbb{P}\left(\bigcap_{i=1}^{n+1} \left\{ \left\| Z^{(m)}_{\varepsilon_{i,k}}; Z^{(m)} \right\|_{Q_i} \to 0 \right\} \right) =$ 1.
3. Take $\varepsilon_k := \varepsilon_{k,k}$.

Let $\Omega_m := \{\forall \text{ box } Q : \left\| Z^{(m)}_{\varepsilon_k}; Z^{(m)} \right\|_Q \to 0 \quad \text{as } k \to \infty\}$. Since $(\varepsilon_k)_{k\geq n}$ is a subsequence of $(\varepsilon_{n,k})_{k\geq n}$ we obtain

$$\begin{aligned}
\mathbb{P}(\Omega_m^\complement) &= \mathbb{P}\left(\exists Q : \left\| Z^{(m)}_{\varepsilon_k}; Z^{(m)} \right\|_Q \not\to 0 \quad \text{as } k \to \infty \right) \\
&= \mathbb{P}\left(\exists n \geq 1 : \left\| Z^{(m)}_{\varepsilon_k}; Z^{(m)} \right\|_{Q_n} \not\to 0 \quad \text{as } k \to \infty \right) \\
&\leq \mathbb{P}\left(\exists n \geq 1 : \left\| Z^{(m)}_{\varepsilon_{n,k}}; Z^{(m)} \right\|_{Q_n} \not\to 0 \quad \text{as } k \to \infty \right) = 0 ,
\end{aligned}$$

II.3.2 Construction of the operator

The operator $\mathcal{H}(Q_L, \beta\xi)$ will be defined as the limit of the operators $-\Delta + \beta\xi_\varepsilon + C_\varepsilon(\beta)$ as $\varepsilon \downarrow 0$, where $C_\varepsilon(\beta)$ is the renormalisation constant introduced in Appendix II.A.

The limit will be taken in the norm resolvent sense: for all a large enough, the operator $(-\Delta+\beta\xi_\varepsilon+C_\varepsilon(\beta)+a)^{-1}$ converges in norm to some limit that we denote $(-\Delta+\beta\xi+a)^{-1}$. This limit happens to be invertible and self-adjoint, and the operator $-\Delta+\beta\xi$ can then be defined by inversion.

Recall that $\mathscr{M}_{m,Q}$ is the space of all admissible models associated with the convolution kernel $P_+^{(m)}$ and restricted to Q. For any constant $K > 0$ we define the following subset

$$\mathscr{M}_{m,Q,K} := \{Z \in \mathscr{M}_{m,Q} : \|\!|Z; Z(0)|\!\|_Q \le K\} \;,$$

where $Z^{(m)}(0)$ denotes the admissible model associated with the null potential $V \equiv 0$ and the kernel $P_+^{(m)}$. Note that $\mathscr{M}_{m,Q,K}$ is a closed subset of $\mathscr{M}_{m,Q}$.

Proposition II.15. *Fix $m > 0$ and a box Q. There exists a constant $K = K(m) > 0$ and a continuous map $\Phi_{m,Q}$ from $\mathscr{M}_{m,Q,K} \times (-2,2)$ into the set of bounded operators on $L^2(Q)$ endowed with the operator norm topology, such that:*

1. *for any $V \in L^2(Q)$, if the canonical model $Z(V)$ belongs to $\mathscr{M}_{m,Q,K}$ then for any $b \in (-2,2)$ we have $\Phi_{m,Q}(Z,b) = (-\Delta + V + m + b)^{-1}$,*
2. *if the renormalised model $Z_\varepsilon^{(m)}(\beta\xi_\varepsilon)$ belongs to $\mathscr{M}_{m,Q,K}$, then for any $b \in (-2,2)$ we have $\Phi_{m,Q}(Z_\varepsilon^{(m)}, b) = (-\Delta + \beta\xi_\varepsilon + C_\varepsilon(\beta) + m + b)^{-1}$.*

Moreover, the constant $K(m)$ increases to infinity with m.

Proof. This is a consequence of the fixed point result [Lab19, Prop. 3.7], combined with the reconstruction theorem [Lab19, Thm. 3] and some computations as in [Lab19, Lem 4.5]. □

Proof of Proposition II.3. Fix $\beta > 0$. Properties 2. and 3. come from standard applications of the min-max formula when the potential is in $L^2(Q)$, in particular for $\beta\xi_\varepsilon+C_\varepsilon(\beta)$. Consequently, on the event where 1. holds, one can pass to the limit on ε_k and deduce 2. and 3. We are therefore left with the proof of Property 1.

Let us introduce the event

$$\tilde\Omega_{m,n} := \{Z^{(m)}(\beta\xi) \in \mathscr{M}_{m,Q_n,\frac{1}{2}K(m)}\} \;.$$

It was proven[1] in [Lab19, Prop 4.1] that for any given $n \ge 1$ and any given constant $C > 0$, we have $\mathbb{P}(Z^{(m)}(\beta\xi) \in \mathscr{M}_{m,Q,K}) \to 1$ as $m \to \infty$ sufficiently fast so that $\mathbb{P}(\liminf_{m\to\infty} \tilde\Omega_{m,n}) = 1$ by Borel-Cantelli. Consequently, the event

$$\tilde\Omega := \bigcap_{n\ge1} \liminf_{m\to\infty} \tilde\Omega_{m,n} \;,$$

has probability one.

[1]The proof is presented for $\beta = 1$ in that reference but the arguments are exactly the same for a general β.

We now argue deterministically on the event $\Omega' \cap \tilde{\Omega}$ of probability one. For any box Q, taking $n \geq 1$ such that $Q \subset Q_n$, there exists $m_0 \geq 1$ such that for all $m \geq m_0$

$$Z^{(m)} \in \mathscr{M}_{m,Q_n,\frac{1}{2}K(m)} \subset \mathscr{M}_{m,Q,\frac{1}{2}K(m)} ,$$

and

$$\left\|\!\left| Z^{(m)}_{\varepsilon_k}; Z^{(m)} \right|\!\right\|_Q \to 0 , \quad k \to \infty .$$

Consequently for all k large enough we have

$$Z^{(m)}_{\varepsilon_k} \in \mathscr{M}_{m,Q_n,K(m)} \subset \mathscr{M}_{m,Q,K(m)} .$$

Note that for all $a > m_0 - 2$, we can always find a pair $(m, b) \in \mathbb{N} \times (-2, 2)$ such that $a = m + b$ and $m \geq m_0$. For all such a, we define

$$\mathcal{G}_a := \Phi_{m,Q}(Z^{(m)}, b) , \quad \mathcal{G}_a^{\varepsilon_k} := \Phi_{m,Q}(Z^{(m)}_{\varepsilon_k}, b) .$$

Proposition II.15 thus ensures that $\mathcal{G}_a^{\varepsilon_k}$ converges in norm to $\mathcal{G}_a$ as $k \to \infty$. Note that this definition of $\mathcal{G}_a$ does not depend on the choice of the pair (m, b) made above: indeed, choosing some alternative parameters (m', b') we have

$$\begin{aligned}\Phi_{m',Q}(Z^{(m')}_{\varepsilon_k}, b') &= (-\Delta + \beta\xi_{\varepsilon_k} + m' + C_{\varepsilon_k}(\beta) + b')^{-1} \\ &= (-\Delta + \beta\xi_{\varepsilon_k} + m + C_{\varepsilon_k}(\beta) + b)^{-1} = \Phi_{m,Q}(Z^{(m)}_{\varepsilon_k}, b) ,\end{aligned}$$

so that by passing to the limit on k we get $\Phi_{m,Q}(Z^{(m)}, b) = \Phi_{m',Q}(Z^{(m')}, b')$.
The arguments in [Lab19, Prop 4.2] finally show that there exists a self-adjoint operator $\mathcal{H}(Q, \beta\xi)$ with pure point spectrum bounded from below whose resolvent is given by $(\mathcal{H}(Q, \beta\xi) + a)^{-1} = \mathcal{G}_a$. In addition, $\mathcal{G}_a^{\varepsilon_k} = (\mathcal{H}(Q, \beta\xi_{\varepsilon_k}) + C_{\varepsilon_k}(\beta)) + a)^{-1}$ converges in norm to $(\mathcal{H}(Q, \beta\xi) + a)^{-1}$ thus concluding the proof. □

II.3.3 Proof of Proposition II.8

We now turn to the proof of the large deviations estimate of Proposition II.8. From now on, we work on the box Q_L for some $L \geq 1$ fixed once and for all.

The proof relies on the following result of Hairer and Weber. We would like to point out that, although this result is originally spelled out in the regularity structure built for the dynamical Φ^4_3 model, the proof relies on a large deviations principle for Wiener chaos, and, as mentioned on [HW15, p.59], it is applicable to all equations that can be treated with the theory of regularity structures, in particular to our case.

Theorem II.16 (Theorem 4.3 in [HW15])**.** *Fix $m \geq 1$. The collection $(Z^{(m)}(\beta\xi))_{\beta\in(0,1]}$ of admissible models in $\mathscr{M}_{m,Q_L}$ satisfies a large deviation principle with rate β^2 and rate function*

$$I_{\mathscr{M}}(Z) := \inf\left\{\frac{1}{2}\|V\|^2_{L^2(Q_L)} : Z^{(m)}(V) = Z\right\}$$

where $Z^{(m)}(V)$ is the canonical model associated to the deterministic noise V.

For $m \geq 1$ introduce the event $E_{m,L}(\beta)$ on which the operator $\Phi_{m,Q_L}(Z^{(m)}(\beta\xi), 0)$ is well-defined and positive, i.e. $E_{m,L}(\beta)$ is the event on which

1. the model $Z^{(m)}(\beta\xi)$ belongs to $\mathscr{M}_{m,Q_L,K(m)}$, where $K(m)$ is the constant of Proposition II.15,
2. $\Phi_{m,Q_L}(Z^{(m)}(\beta\xi), 0)$ is a positive operator on $L^2(Q_L)$.

The event $E_{m,L}(\beta)$ occurs with high probability, as shown by the following estimate.

Lemma II.17. *There exist some constants $\alpha, M, \nu > 0$ such that for all $m \geq 1$ large enough, all $L \geq 1$ and all $\beta \in (0, 1]$ we have*

$$\mathbb{P}\left((E_{m,L}(\beta))^{\complement}\right) \leq ML^d e^{-\frac{\alpha m^\nu}{\beta^2}} .$$

Proof of Lemma II.17. Suppose that $Z^{(m)} = Z^{(m)}(\beta\xi) \in \mathscr{M}_{m,L,\frac{1}{2}K(m)}$, then the arguments in the proof of Proposition II.3 combined with Proposition II.15 show that $(\mathcal{H}(Q, \beta\xi) + a)$ is invertible for all $a \in (m-2, m+2)$. We thus deduce that if for every $m \geq m_0$, we have $Z^{(m)} \in \mathscr{M}_{m,L,\frac{1}{2}K(m)}$, then $(\mathcal{H}(Q, \beta\xi) + a)$ is invertible for all $a \in (m_0 - 2, +\infty)$, thus implying that all these operators $(\mathcal{H}(Q, \beta\xi) + a)^{-1}$ are positive, in particular $\Phi_{m,Q_L}(Z^{(m)}, 0)$.
Now let $m_0 \in \mathbb{N}$ be arbitrary. We deduce that

$$\mathbb{P}\left((E_{m_0,L})^{\complement}\right) \leq \mathbb{P}\left(\sup_{m \geq m_0} \left\| Z^{(m)}; Z(0) \right\|_{Q_L} \geq \frac{K(m)}{2}\right) \leq \sum_{m \geq m_0} \mathbb{P}\left(\left\| Z^{(m)}; Z(0) \right\|_{Q_L} \geq \frac{K(m)}{2}\right) .$$

Write $Z^{(m)}(\beta\xi) = (\Pi^{(m),\beta}, \Gamma^{(m),\beta})$. By [Lab19, Lemma 2.3 and Lemma 3.1], there exists a constant $C > 0$ such that for all $m \geq 1$ we have

$$\mathbb{P}\left(\left\| Z^{(m)}; Z(0) \right\|_{Q_L} \geq \frac{K(m)}{2}\right) \leq \mathbb{P}\left(\left\| \Pi^{(m),\beta} \right\|_{\Lambda,Q_L} \geq \frac{CK(m)}{2}\right) ,$$

where

$$\left\| \Pi^{(m),\beta} \right\|_{\Lambda,Q_L} := \sup_{\zeta \in \mathcal{A}_{<0}} \sup_{\tau \in \mathcal{A}_\zeta} \left\| \Pi^{(m),\beta} \tau \right\|_{\Lambda,Q_L} ,$$

and, for some scaling function φ of a compactly supported wavelet basis

$$\left\| \Pi^{(m),\beta} \tau \right\|_{\Lambda,Q_L} := \sup_{n \geq n_m} \sup_{x \in Q_L \cap (2^{-n}\mathbb{Z}^d)} \frac{\left| \left\langle \Pi_x^{(m),\beta} \tau, \varphi_x^{2^{-n}} \right\rangle \right|}{\|\tau\|_\zeta 2^{-n\zeta}} .$$

Let $\Pi^{(m)} = \Pi^{(m),1}$ be the model associated to the noise ξ with scaling factor $\beta = 1$. We note that $\Pi_x^{(m),\beta}\tau = \beta^{\|\tau\|}\Pi_x^{(m)}\tau$, where $\|\tau\|$ is the number of occurrences of Ξ in the symbol τ. By [Lab19, Lemma 4.11] there exist $\lambda, \nu > 0$ such that

$$M := \sup_{m \geq 1} \sup_{L \geq 1} \frac{1}{L^d} \sum_{\tau \in \mathcal{A}_{<0}} \mathbb{E}\left[\exp\left(\lambda m^\nu \left\| \Pi^{(m)} \tau \right\|_{\Lambda,Q_L}^{\frac{2}{\|\tau\|}}\right)\right] < \infty .$$

A simple computation thus yields the existence of a constant $\alpha > 0$ such that for all $m \geq 1$ and all $L \geq 1$

$$\mathbb{P}\left(\left\|\Pi^{(m),\beta}\right\|_{\Lambda,Q_L} \geq \frac{1}{2}CK(m)\right) \leq ML^d e^{-\alpha\frac{m^\nu}{\beta^2}} ,$$

which concludes the proof upon summing for $m \geq m_0$. □

Proof of Proposition II.8. Fix any $m \geq 1$. On the event $E_{m,L}(\beta) \cap \Omega' \cap \tilde{\Omega}$ (recall the proof of Proposition II.3), the operator $\Phi_{m,Q_L}(Z^{(m)}(\beta), 0)$ coincides with the operator $(\mathcal{H}(Q_L, \beta\xi) + m)^{-1}$; it is positive, self-adjoint and compact, implying that its spectrum consists of positive eigenvalues that we denote $(\mu_{n,L})_{n\geq 1}$ in non-increasing order. These eigenvalues can be related to those of $\mathcal{H}(Q_L, \beta\xi)$, that we denote $(\lambda_{n,L})_{n\geq 1}$ in non-decreasing order, through:

$$\lambda_{n,L} = (\mu_{n,L})^{-1} - m .$$

Fix some $c \in \mathbb{R}$ and observe that $I_L((-\infty, c])$ and $I_L((-\infty, c))$ are finite.

Upper bound. Recall the constant $K = K(m)$ of Proposition II.15. On the set $\mathscr{M}_{m,Q_L,K}$, we define the continuous mapping $\varphi_m : \mathscr{M}_{m,Q_L,K} \to \mathbb{R}$ by $Z \mapsto \mu$ where μ is the supremum of the spectrum of the operator $\Phi_{m,L}(Z, 0)$. The continuity of φ_m is a consequence of the continuity of $\Phi_{m,L}$ and of the variational formula for the supremum of the spectrum. Take m large enough such that $m + c > 0$, we can write[2]

$$\begin{aligned}\mathbb{P}\left(\lambda_{1,L} \leq c; E_{m,L}(\beta)\right) &= \mathbb{P}\left(\mu_{1,L} \geq \frac{1}{m+c}; E_{m,L}(\beta)\right)\\ &= \mathbb{P}\left(Z^{(m)}(\beta\xi) \in \varphi_m^{-1}\Big(\big[\frac{1}{m+c}, \infty\big)\Big); E_{m,L}(\beta)\right) .\end{aligned}$$

Therefore

$$\mathbb{P}(\lambda_{1,L} \leq c) \leq \mathbb{P}\left(Z^{(m)}(\beta\xi) \in \varphi_m^{-1}\big([\frac{1}{m+c}, \infty)\big)\right) + \mathbb{P}\left(E_{m,L}(\beta)^{\complement}\right) .$$

Assume that we can show that for all $m \geq 1$, we have

$$\limsup_{\beta\to 0} \beta^2 \log \mathbb{P}\left(Z^{(m)}(\beta\xi) \in \varphi_m^{-1}\big([\frac{1}{m+c}, \infty)\big)\right) \leq -I_L((-\infty, c]) . \tag{II.23}$$

Then by Lemma II.17 we deduce that

$$\limsup_{\beta\to 0} \beta^2 \log \mathbb{P}(\lambda_{1,L} \leq c) \leq \max\left\{-I_L((-\infty, c]), -cm^\nu\right\} .$$

Choosing m large enough so that $\alpha m^\nu > I_L((-\infty, c])$, we obtain the desired upper bound of the large deviation estimate. We are left with proving (II.23).

[2]Implicitly any subset of $\mathscr{M}_{m,Q_L,K}$ is viewed as a subset of $\mathscr{M}_{m,Q_L}$: in particular, $\varphi_m^{-1}(A)$ for any set $A \subset \mathbb{R}$.

Since $\mathscr{M}_{m,Q_L,K}$ is a closed subset of $\mathscr{M}_{m,Q_L}$, $\varphi_m^{-1}\big([\frac{1}{m+c},\infty)\big)$ is itself a closed subset of $\mathscr{M}_{m,Q_L}$. The large deviations principle stated in Theorem II.16 thus yields

$$\limsup_{\beta\to 0} \beta^2 \log \mathbb{P}\left(Z^{(m)}(\beta\xi) \in \varphi_m^{-1}\big([\frac{1}{m+c},\infty)\big)\right) \le -I_{\mathscr{M}}(\varphi_m^{-1}([\frac{1}{m+c},\infty)))\ .$$

At this point, observe that for any $V \in L^2(Q_L)$ such that $Z^{(m)}(V) \in \mathscr{M}_{m,Q_L,K}$, the operator $(-\Delta + V + m)^{-1}$ is well-defined and $\mu = \varphi_m(Z^{(m)}(V))$ is its largest positive eigenvalue. A priori we do not know whether this operator is positive so that μ does not necessarily correspond to the smallest eigenvalue $\lambda_1(Q_L, V)$ of $-\Delta + V$ and we only have the inequality

$$\lambda_1(Q_L, V) \le \frac{1}{\mu} - m\ .$$

As a consequence we find

$$\begin{aligned} I_{\mathscr{M}}\left(\varphi_m^{-1}([\frac{1}{m+c},\infty))\right) &= \inf\left\{\frac{\|V\|^2_{L^2(Q_L)}}{2} : Z^{(m)}(V) \in \mathscr{M}_{m,Q_L,K}, \frac{1}{\mu} - m \le c\right\} \\ &\ge \inf\left\{\frac{\|V\|^2_{L^2(Q_L)}}{2} : \lambda_1(Q_L, V) \le c\right\}\ . \end{aligned}$$

The above implies

$$\limsup_{\beta\to 0} \beta^2 \log \mathbb{P}(\lambda_{1,L} \le c) \le -I_L((-\infty, c])\ ,$$

as required.

Lower bound. We consider the *open* subset of $\mathscr{M}_{m,Q_L}$ defined by

$$\mathscr{M}_{m,Q_L,K-} := \left\{Z \in \mathscr{M}_{m,Q_L} : \left\|Z; Z^{(m)}(0)\right\|_{Q_L} < K\right\}\ .$$

with the constant $K = K(m)$ of Proposition II.15. On the set $\mathscr{M}_{m,Q_L,K-}$, we consider the continuous mapping $\varphi_m : \mathscr{M}_{m,Q_L,K-} \to \mathbb{R}$ by $Z \mapsto \mu$ where μ is the supremum of the spectrum of the operator $\Phi_{m,L}(Z, 0)$. Here again φ_m is continuous. Again take m large enough such that $m + c > 0$. From similar arguments as before we have

$$\mathbb{P}(\lambda_{1,L} < c) \ge \mathbb{P}\left(Z^{(m)}(\beta\xi) \in \varphi_m^{-1}\big((\frac{1}{m+c},\infty)\big)\right) - \mathbb{P}\left(E_{m,L}(\beta)^{\complement}\right)\ .$$

Note that $\varphi_m^{-1}((\frac{1}{m+c},\infty))$ is an open set. Therefore the large deviations principle stated in Theorem II.16 yields

$$\liminf_{\beta\to 0} \beta^2 \log \mathbb{P}\left(Z^{(m)}(\beta\xi) \in \varphi_m^{-1}\big((\frac{1}{m+c},\infty)\big)\right) \ge -I_{\mathscr{M}}\left(\varphi_m^{-1}\big((\frac{1}{m+c},\infty)\big)\right)\ .$$

Note that

$$\inf I_{\mathscr{M}}\left(\varphi_m^{-1}\Big((\frac{1}{m+c},\infty)\Big)\right) = \inf\left\{\frac{\|V\|^2_{L^2(Q_L)}}{2} : Z^{(m)}(V)\in\mathscr{M}_{m,Q_L,K_-}, \frac{1}{\mu}-m<c\right\} .$$

We would like to compare this quantity to $I_L\left((-\infty,c)\right)$: here again the operator $(-\Delta+V+m)^{-1}$ may not be positive so that the supremum of its spectrum may not be related to $\lambda_1(Q_L,V)$.

Recall that $I_L((-\infty,c))<\infty$ so for any given $\delta>0$, there exists some function $V\in L^2$ such that $\lambda(Q_L,V)\in(-\infty,c)$ and such that $\|V\|^2_{L^2}/2\le\inf I_L((-\infty,c))+\delta$. Recall from Proposition II.15 that the constant $K(m)$ increases to infinity as $m\to\infty$; hence for this particular V, by (II.24) we can find m sufficiently large such that $Z^{(m)}(V)\in\mathscr{M}_{m,Q_L,K}$ and such that $\lambda_1(Q_L,V)+m>0$. This implies $1/\mu-m=\lambda_1(Q_L,V)\in(-\infty,c)$ and thus proves that for m large enough

$$I_{\mathscr{M}}\left(\varphi_m^{-1}\Big((\frac{1}{m+c},\infty)\Big)\right)\le I_L((-\infty,c))+\delta .$$

On the other hand, by Lemma II.17 we know that, provided m is large enough, $\mathbb{P}\left(E_{m,L}(\beta)^{\complement}\right)$ is negligible compared to the term that we have just controlled and therefore

$$\liminf_{\beta}\beta^2\log\mathbb{P}(\lambda<c)\ge -I_L((-\infty,c))-\delta ,$$

for any given $\delta>0$. This suffices to conclude. □

II.A RENORMALISATION CONSTANTS

We let G be the Green's function of $-\Delta$, and $P^{(m)}$ be the Green's function of $-\Delta+m$, we refer to [Lab19, Sec 3.1] for the expressions. Recall that n_m is the smallest integer such that $2^{-n_m}\le 1/\sqrt{m}$.

Lemma II.18. *Fix $r>0$. For every $m\ge 1$, there exists a decomposition $P^{(m)}=P_+^{(m)}+P_-^{(m)}$ such that:*

1. *$P_+^{(m)}$ is supported in $B(0,2^{-n_m+1})$ and satisfies $P_+^{(m)}=P^{(m)}$ on $B(0,2^{-n_m-1})$, while $P_-^{(m)}$ is $\mathcal{C}^\infty$ and vanishes on $B(0,2^{-n_m-1})$.*
2. *For all $k\in\mathbb{N}^d$ such that $|k|<r$ we have $\int x^kP_+^{(m)}(x)dx=0$.*
3. *There exists a constant $C>0$, independent of $m\ge 1$, such that for all $k\in\mathbb{N}^d$ such that $|k|<r$ we have*

$$|\partial^kP_+^{(m)}(x)|\le C|\partial^kG(x)| ,\quad x\in\mathbb{R}^d .$$

Proof. This is a consequence of [Lab19, Lemma 3.1] except for the property "$P_+^{(m)} = P^{(m)}$ on $B(0, 2^{-n_m-1})$" that was not stated there. However, this property follows if one picks carefully the functions η_k in that proof: namely, it suffices to impose to the functions η_k to be supported in $B(0,1)\backslash B(0,1/2)$. This can always be achieved, see for instance [CZ20, Lemma 8.1]. □

We introduce the renormalisation constants as follows. In dimension 1, we set $C_\varepsilon^{(m)}(\beta) := 0$. In dimension 2, we set

$$C_\varepsilon^{(m)}(\beta) := \beta^2 \int_{\mathbb{R}^2} P_+^{(m)}(x)\rho_\varepsilon^{*2}(x)\,\mathrm{d}x.$$

A computation shows that there exists a constant $\tilde{c}_\rho(m)$ independent of β such that $C_\varepsilon^{(m)}(\beta) = \beta^2(2\pi)^{-1}\ln\varepsilon^{-1} + \beta^2\tilde{c}_\rho(m) + o(1)$ as $\varepsilon \downarrow 0$.
In dimension 3, we set $C_\varepsilon^{(m)}(\beta) := \beta^2 c_\varepsilon^{(m)} + \beta^4 c_\varepsilon^{(m),1,1} + \beta^4 c_\varepsilon^{(m),1,2}$ where

$$\begin{aligned} c_\epsilon^{(m)} &:= \int P_+^{(m)}(x)\rho_\varepsilon^{*2}(x)\,\mathrm{d}x, \\ c_\epsilon^{(m),1,1} &:= \iiint P_+^{(m)}(x_1)P_+^{(m)}(x_2)P_+^{(m)}(x_3)\rho_\varepsilon^{*2}(x_1+x_2)\rho_\varepsilon^{*2}(x_2+x_3)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3, \\ c_\epsilon^{(m),1,2} &:= \iiint P_+^{(m)}(x_1)P_+^{(m)}(x_2)\Big(P_+^{(m)}(x_3)\rho_\varepsilon^{*2}(x_3) - c_\epsilon\delta_0(x_3)\Big)\rho_\varepsilon^{*2}(x_1+x_2+x_3)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3. \end{aligned}$$

There exist some constants $c_\rho, \tilde{c}_\rho, \tilde{c}_\rho^{1,1}(m), c_\rho^{1,2}$ independent of β such that as $\varepsilon \downarrow 0$

$$\begin{aligned} c_\epsilon^{(m)} &= \frac{c_\rho}{\varepsilon} + \tilde{c}_\rho\sqrt{m} + o(1)\ , \\ c_\epsilon^{(m),1,1} &= \ln\frac{1}{\varepsilon} + \tilde{c}_\rho^{1,1}(m) + o(1)\ , \\ c_\epsilon^{(m),1,2} &= c_\rho^{1,2} + o(1)\ . \end{aligned}$$

Note that the only constant that depends on m is $\tilde{c}_\rho^{1,1}(m)$, and its expression is a bit involved so we refrain from writing it explicitly. On the other hand, if we let $G(x) = \frac{1}{4\pi|x|}$ (which is nothing but the Green's function of $-\Delta$), we have $c_\rho = \int_{\mathbb{R}^3} G(y)\rho^{*2}(y)dy$, $\tilde{c}_\rho = -\int_{\mathbb{R}^3} G(y)|y|\rho^{*2}(y)dy$, and

$$c_\rho^{1,2} = \iiint G(y_1)G(y_3)\left(G(z_2-y_3) - G(z_2) - \langle\nabla G(z_2), y_3\rangle\right)\rho^{*2}(y_3)\rho^{*2}(y_1+z_2)\,\mathrm{d}y_1\,\mathrm{d}z_2\,\mathrm{d}y_3.$$

The construction of the renormalised model $Z_\varepsilon^{(m)}(\beta)$ follows along the lines of [HP15] and [Lab19]. However, we take slightly different renormalisation constants compared to [Lab19]: instead of taking the constants built from the kernel $P_+^{(m)}$, we take those associated to $P_+^{(1)}$. Namely, in dimension 2, we take $C_\varepsilon := C_\varepsilon^{(1)}$ and in dimension 3, we take the three constants $c_\varepsilon^{(1)}$, $c_\varepsilon^{(1),1,1}$ and $c_\varepsilon^{(1),1,2}$. This produces a limiting renormalised model that differs from the one in [Lab19] by *finite* constants, as shown by the above asymptotics: this does not modify the final operator, but greatly simplify its definition (in particular, one does not need to deal with constants like $C^{(m)-(1)}$ as in [Lab19]).

Let us finally mention that the renormalised model satisfies for $d = 2$

$$\Pi_x^{(m),\varepsilon}(\beta)\Xi\mathcal{I}(\Xi)(x) = -C_\varepsilon(\beta) ,$$

and for $d = 3$

$$\Pi_x^{(m),\varepsilon}(\beta)\Xi\mathcal{I}(\Xi)(x) = -c_\varepsilon^{(1)}(\beta) , \quad \Pi_x^{(m),\varepsilon}(\beta)\Xi\mathcal{I}(\Xi\mathcal{I}(\Xi\mathcal{I}(\Xi)))(x) = -c_\varepsilon^{(1),1,1}(\beta) - c_\varepsilon^{(1),1,2}(\beta) .$$

II.B CANONICAL MODEL

The aim of this section is to construct a canonical admissible model $Z^{(m)}(V) := (\Pi_x^{(m)}(V), \Gamma_{xy}^{(m)}(V))$ associated to some potential function $V \in L^2_{\text{loc}}(\mathbb{R}^d)$ and to show that

$$\sup_{m\geq 1} \left\| Z^{(m)}(V); Z^{(m)}(0) \right\|_Q < \infty . \tag{II.24}$$

for all box $Q \subset \mathbb{R}^d$. If V were smooth, then [Hai14, Prop. 8.27] would ensure that the model is admissible. Moreover, once the model is defined, (II.24) essentially follows from Lemma II.18 since the kernel $P_+^{(m)}$ is controlled by G uniformly over all m. However, here V is only in $L^2_{\text{loc}}(\mathbb{R}^d)$, which necessitates some adjustments in order to obtain the required analytical bounds. In the sequel, we fix $V \in L^2_{\text{loc}}(\mathbb{R}^d)$.

The set of symbols $\mathcal{T} = \mathcal{U} \cup \mathcal{F}$ introduced in Subsection II.3.1 can be obtained through a recursive construction: let $\mathcal{U}_0 = \{\mathbf{1}, X^k\}$, and for $n \geq 0$ we define recursively

$$\begin{aligned} \mathcal{F}_n &:= \{\Xi\tau : \tau \in \mathcal{U}_n\} , \\ \mathcal{U}_{n+1} &:= \{\mathbf{1}, X^k\} \cup \{\mathcal{I}(\tau) : \tau \in \mathcal{F}_n, |\tau| + 2 < \gamma\} , \end{aligned}$$

for $\gamma = 2 - 4\kappa$. Subsequently we have $\mathcal{F} = \bigcup_{n\geq 0} \mathcal{F}_n$ and $\mathcal{U} = \bigcup_{n\geq 0} \mathcal{U}_n$. Note that all elements in $\mathcal{U}$ are of positive homogeneity.

Given this recursive structure, we can define the model $\Pi^{(m)}$ in the following manner: for $m \geq 1$, $x, y \in \mathbb{R}^d$, we set

$$\Pi_x^{(m)}\mathbf{1}(y) = 1, \quad \Pi_x^{(m)} X^k(y) = (y - x)^k, \quad \Pi_x^{(m)}\Xi(y) = V(y)$$

and then recursively

$$\begin{aligned} &\Pi_x^{(m)}(\tau\Xi)(y) = (\Pi_x^{(m)}\tau)(y) \cdot V(y), \\ &\Pi_x^{(m)}(\mathcal{I}\bar\tau)(y) = \int P_+^{(m)}(y - z)\Pi_x^{(m)}\bar\tau(z)dz - \sum_{|k|<|\bar\tau|+2} \frac{(y - x)^k}{k!} \int D^k P_+^{(m)}(x - z)\Pi_x^{(m)}\bar\tau(z)dz \end{aligned}$$

for all $\tau \in \mathcal{U}$ and $\bar\tau \in \mathcal{F}$. Since V is a function, all these expressions are well-defined.

We need the following estimate, that follows from standard arguments based on Lemma II.18. If f_x is a function that satisfies

$$|f_x(y)| \lesssim |x - y|^\zeta ,$$

uniformly over all $y \in \mathbb{R}^d$ such that $|x-y| \le C$, then

$$\left| \int P_+^{(m)}(y-z)V(z)f_x(z)dz - \sum_{|k|<|\tau|+2} \frac{(y-x)^k}{k!} \int D^k P_+^{(m)}(x-z)V(z)f_x(z)dz \right| \lesssim |x-y|^{\zeta+2-\frac{d}{2}} ,$$

uniformly over all $y \in \mathbb{R}^d$ such that $|x-y| \le C-1$ and over all $m \ge 1$.

Let us now prove recursively the required analytical bounds on $\Pi^{(m)}$ for some fixed box Q. Pick a large constant $C > 0$. Suppose that for all $\tau \in \mathcal{U}_n$, we have $|\Pi_x^{(m)}\tau(y)| \lesssim |y-x|^{|\tau|}$ uniformly over all $x \in Q$ and all $y \in \mathbb{R}^d$ such that $|x-y| \le C$ and over all $m \ge 1$. For any $\tau \in \mathcal{U}_n$ we have, by the Cauchy-Schwarz inequality

$$\left|\langle \Pi_x^{(m)}\Xi\tau, \varphi_x^\lambda \rangle\right| \lesssim \lambda^{|\tau|} \int |V(y)||\varphi_x^\lambda(y)|dy \le \lambda^{|\tau|} \|V\,\mathbf{1}_{B(x,C)}\|_{L^2} \lambda^{-\frac{d}{2}} \lesssim \lambda^{|\tau\Xi|} ,$$

uniformly over all $x \in Q$, all $m \ge 1$, all $\varphi \in \mathscr{B}^r$ and al $\lambda \in (0, C]$. Furthermore, by the estimate above

$$\left|\Pi_x \mathcal{I}(\Xi\tau)(y)\right| \le |x-y|^{|\tau|+2-\frac{d}{2}} ,$$

uniformly over all $x \in Q$ and all $y \in \mathbb{R}^d$ such that $|x-y| \le C-1$ and over all $m \ge 1$.

Since only finitely many iterations suffice to exhaust the whole set $\mathcal{U} \cup \mathcal{F}$, we deduce that

$$\left|\left\langle \Pi_x^{(m)}\tau, \varphi_x^\lambda \right\rangle\right| \lesssim \lambda^{|\tau|}$$

uniformly over all $x \in Q$, all $m \ge 1$, all $\varphi \in \mathscr{B}^r$, all $\lambda \in (0,1]$ and all $\tau \in \mathcal{U} \cup \mathcal{F}$.

Regarding the construction of $\Gamma^{(m)}$, we argue that it is uniquely determined once $\Pi^{(m)}$ is specified on the negative levels of the regularity structure, see for example [HW15, Thm. 2.10]. Now that the model $Z^{(m)}(V)$ is defined with respect to $V \in L^2_{\text{loc}}$ and that the bound on $\left\|\Pi^{(m)}\right\|_Q$ is independent of m, we can invoke [Lab19, Lem. 2.3] to conclude $\left\|Z^{(m)}(V); Z^{(m)}(0)\right\|_Q$ is bounded by a constant independent of m, whence (II.24) follows.

II.C PROOF OF PROPOSITION II.7

The goal of this subsection is to prove Proposition II.7, which is basically an extension of a result by Gärtner and König [GK00] where they have considered the case of smooth bounded potential. We begin by proving a variation to their Proposition 1 in [GK00].

Proposition II.19. *For fixed $L > r > 0$ and any bounded smooth potential V, there exists a constant $K > 0$ such that*

$$\lambda(Q_L, V) \ge \min_{k \in \mathbb{Z}^d : |k|_\infty \le \frac{L}{2r} + \frac{3}{4}} \lambda(rk + Q_{\frac{3r}{2}}, V) - \frac{K}{r^2}.$$

Proof. The proof is built upon a specific choice of partition of unity: Let $\eta : \mathbb{R}^d \to [0,1]$ be a smooth function supported in $Q_{\frac{3r}{2}}$ such that it gives 1 on $Q_{r/2}$, $\sum_{k\in\mathbb{Z}^d} \eta_k^2(x) = 1$ and that $\sum_{k\in\mathbb{Z}^d} |\nabla\eta_k|^2(x) \le K/r^2$ for all $x \in \mathbb{R}^d$, where $\eta_k(x) := \eta(rk+x)$. We will give a construction of such partition later in the proof.

Note first that we have the following variational formulation for the principal eigenvalue of the operator $-\Delta + V$ on a domain $D \subset \mathbb{R}^d$:

$$\lambda(D,V) = \inf_{\substack{\psi\in C_c^\infty(D)\\ \|\psi\|_{L^2}=1}} \int_{\mathbb{R}^d} |\nabla\psi|^2 + V\psi^2 =: \inf_{\|\psi\|_{L^2}=1} G^V(\psi).$$

Given the desired partition of unity (η_k), we take $\psi \in C_c^\infty(Q_L)$ such that $\|\psi\|_{L^2} = 1$ and set $\psi_k = \eta_k\psi$. With the fact that $|\nabla\psi_k|^2 = \eta_k^2|\nabla\psi|^2 + \psi^2|\nabla\eta_k|^2 + \nabla(\eta_k^2)\cdot\nabla(\psi^2)/2$, it follows that $\sum_k |\nabla\psi_k|^2 = |\nabla\psi|^2 + \sum_k |\nabla\eta_k|^2\psi^2$. Therefore

$$\sum_{k\in\mathbb{Z}^d} \|\psi_k\|_{L^2}^2 G^V(\psi_k/\|\psi_k\|_{L^2}) = \sum_{k\in\mathbb{Z}^d} \int_{\mathbb{R}^d} \left(|\nabla\psi_k|^2 + V\psi_k^2\right) \le G^V(\psi) + \frac{K}{r^2}$$

where we have used the property $\sup_{x\in\mathbb{R}^d} \sum_k |\nabla\eta_k|^2(x) \le K/r^2$ in the last inequality. Since ψ is supported in Q_L, the sum over k is in fact a finite sum as we can restrict ourselves to those k's such that $r|k|_\infty - 3r/4 < L/2$. Hence

$$G^V(\psi) + \frac{K}{r^2} \ge \sum_{|k|_\infty < \frac{L}{2r}+\frac{3}{4}} \|\psi_k\|_{L^2}^2 \min_{k\in\mathbb{Z}^d:|k|_\infty<\frac{L}{2r}+\frac{3}{4}} \lambda(Q_{kr+\frac{3r}{2}}, V) = \min_{|k|_\infty<\frac{L}{2r}+\frac{3}{4}} \lambda(Q_{kr+\frac{3r}{2}}, V).$$

We then have our desired inequality by taking an infimum over ψ.

Finally, we finish this proof by constructing the function η with desired properties. The d-dimensional construction can be first reduced to a 1-dimensional one by setting $\eta(x) = \zeta(x_1)\ldots\zeta(x_d)$ for $x = (x_1,\ldots,x_d) \in \mathbb{R}^d$ with ζ being the 1-dimensional version of η. On the other hand ζ can be constructed as follows. Let $\varphi(x) := c\int_{-\infty}^x e^{-1/(1-u^2)}\mathbf{1}_{|u|<1}$, where the constant c is chosen so that $\varphi(x) = 1$ for $x \ge 1$. Note that $\varphi(x) = 0$ for $x \le -1$ and that $\varphi(x) + \varphi(-x) = 1$ for all $x \in \mathbb{R}$. One can also verify that $\sqrt{\varphi}$ is smooth. Now set

$$\zeta(x) = \sqrt{\varphi(2(r+2x)/r)\varphi(2(r-2x)/r)}.$$

One can see that $\zeta(x) = 1$ if $|x| \le r/4$, $\zeta(x) = 0$ if $|x| > 3r/4$ and $\sum_k \zeta^2(rk+x) = 1$ for all x. Moreover, since the function φ is independent of r, we have $\|\nabla\eta\|_\infty \lesssim 1/r$ with the proportionality constant depending only on the function φ, thus giving the bound $\sup_{x\in\mathbb{R}^d} \sum_k |\nabla\eta_k|^2(x) \le K/r^2$ for some constant $K > 0$. □

Proof of Proposition II.7. To prove Proposition II.7, we first consider the same assertions with ξ replaced by the mollified and renormalized white noise ξ_ε. In this case, the lower bound of (II.7) follows from Proposition II.19, while the remaining assertions are consequences of the variational formuation of eigenvalues

$$\lambda_n(D,V) := \inf_{\substack{F\sqsubset C_c^\infty(D)\\ \dim(F)=n}} \sup_{\substack{\psi\in F\\ \|\psi\|_{L^2}=1}} G^V(\psi)$$

(where the functional G^V is defined in the proof of Lemma II.19) for any domain $D \subset \mathbb{R}^d$ and bounded smooth potential V.

Assertions being established for all smooth bounded V, now it remains to take $V_\varepsilon = \beta\xi_\varepsilon + C_\varepsilon(\beta)$ and to pass to the limit. By Proposition II.3, the eigenvalues of the renormalized Hamiltonian $\mathcal{H}(Q_L, \beta\xi_{\varepsilon_k} + C_{\varepsilon_k}(\beta))$ converge almost surely to those of $\mathcal{H}(Q_L, \beta\xi)$, which implies immediately the desired almost sure inequality (II.7). $\square$

Chapter III

Construction and spectrum of the Anderson Hamiltonian with white noise potential on $\mathbb{R}^2$ and $\mathbb{R}^3$

This chapter is based on the article [HL24], submitted to *Probability and Mathematical Physics*.

SUMMARY OF THE PRESENT CHAPTER

III.1 INTRODUCTION

This article establishes the rigorous construction and some spectral properties of the random Schrödinger operator

$$\mathcal{H} := -\Delta + \xi\ , \qquad \text{on } \mathbb{R}^d\ ,$$

where $d \in \{2, 3\}$, Δ is the continuous Laplacian and ξ is a white noise on $\mathbb{R}^d$.

Let us mention that the spectral properties of random Schrödinger operators $-\Delta + V$ have received much attention in the mathematical physics community, most notably in the discrete case where Δ is the Laplacian on $\mathbb{Z}^d$ and $V(k), k \in \mathbb{Z}^d$ is a collection of i.i.d. r.v. In particular, the Anderson Localization phenomenon [And58] has given rise to a large literature. The potential considered in the present article arises as a scaling limit of a large class of discrete or continuous random fields, thus motivating the study of the operator $\mathcal{H}$. From a physical perspective, this operator can be seen as an idealized model in a situation where the potential V has very small correlation length.

Since white noise is distribution-valued, the mere definition of the operator $\mathcal{H}$ is delicate. Actually, the operator requires renormalisation by infinite constants and recent breakthroughs [GIP15, Hai14] in the field of stochastic PDE provide the required tools to carry out this task. Let us give a brief description of the renormalisation procedure. One considers the smoothed out noise $\xi_\varepsilon := \xi * \varrho_\varepsilon$ where $\varrho_\varepsilon(x) := \varepsilon^{-d}\varrho(x/\varepsilon)$, $x \in \mathbb{R}^d$, for some even, smooth function ϱ compactly supported in the unit ball of $\mathbb{R}^d$ and integrating to 1. If one replaces ξ by ξ_ε then the definition of the operator falls into the scope of classical results [Kat72, FL74]. However the sequence of operators does not converge as $\varepsilon \downarrow 0$. Instead, one needs to identify carefully a diverging (with ε) constant C_ε such that the collection of self-adjoint operators

$$\mathcal{H}_\varepsilon := -\Delta + \xi_\varepsilon + C_\varepsilon \ ,$$

converges to some limit that we call $\mathcal{H}$. Let us mention that C_ε diverges as $(2\pi)^{-1}\log\varepsilon^{-1}$ in dimension 2, while in dimension 3 it diverges as $c_\varrho\varepsilon^{-1} + \ln\varepsilon^{-1}$ for some constant $c_\varrho > 0$.

This programme has been carried out in finite volume (a torus, a bounded box) in dimensions 2 and 3 by several authors [AC15, GUZ20, Lab19, Cv21, Mou21, Mv22, BDM23]. The convergence of $\mathcal{H}_\varepsilon$ towards $\mathcal{H}$ then holds in the norm resolvent sense, in probability. Let us point out that in finite volume, the operator is bounded from below so that its resolvent set admits some real values and fixed point argument can be applied to build the associated resolvent operators.
It turns out that in infinite volume the construction is much harder. The main reason is that the operator is no longer bounded from below (see for instance the large volume asymptotics on the ground state [Cv21, HL22]) so that the above stragegy of proof is no longer applicable.
In a recent work, Ugurcan [Ugu22] proposed a construction of $\mathcal{H}$ on $\mathbb{R}^2$ by adapting a celebrated criterion [FL74] of Faris and Lavine ensuring essential self-adjointness for Schrödinger operators. Let us however point out that Ugurcan does not address the convergence of $\mathcal{H}_\varepsilon$ towards $\mathcal{H}$. More recently, Ueki [Uek23] proposed another construction of $\mathcal{H}$ on $\mathbb{R}^2$, based on the heat semigroup approach of [Mou21]: therein, it is proven that $\mathcal{H}_\varepsilon$ converges in the strong resolvent sense towards $\mathcal{H}$ and that the spectrum of $\mathcal{H}$ is almost surely equal to $\mathbb{R}$.
On the other hand, the construction of $\mathcal{H}$ on $\mathbb{R}^3$ has not been addressed so far.

The two aforementionned works manage to identify the domain of the random operator $\mathcal{H}$ on $\mathbb{R}^2$ using the paracontrolled calculus. Let us point out that these works are technically quite involved. In the present article, we present a relatively simple construction of $\mathcal{H}$, in both dimensions 2 and 3, which is based on the solution theory of the parabolic Anderson model (PAM)

$$\begin{cases} \partial_t u = \Delta u - u\xi \ , & \text{on } \mathbb{R}^d \ , \\ u(t=0,\cdot) = f(\cdot) \ . \end{cases} \tag{III.1}$$

The construction of this PDE is a priori less delicate than the construction of the self-adjoint operator $\mathcal{H}$: indeed, solution theory for PDE is flexible as one simply has to identify a (reasonable) space in which existence and uniqueness holds, while self-adjointness is a delicate property that requires to identify a (random) dense subset of L^2 on which the operator acts. Let us point out that the construction of the (PAM) was established in [HL15, HL18] respectively on $\mathbb{R}^2$ and $\mathbb{R}^3$.
We define the operator $\mathcal{H}$ as the generator of the semigroup associated to the collection of solutions to this PDE. This is the content of our first result:

Theorem III.1. *In dimensions* 2 *and* 3*, there exists a random operator $\mathcal{H}$ which is self-adjoint on (a dense subset of) $L^2(\mathbb{R}^d, dx)$ and which is the limit in probability of $\mathcal{H}_\epsilon$ as $\epsilon \downarrow 0$ in the strong resolvent sense. For any $t \geq 0$, the domain of the operator $e^{-t\mathcal{H}}$ contains all functions $f \in L^2(\mathbb{R}^d, dx)$ with compact support and $e^{-t\mathcal{H}} f$ coincides with the solution of* (III.1) *at time t, starting from f at time* 0*.*

The proof of Theorem III.1 is split into three steps:

$$\begin{matrix} \text{Noise} & \xrightarrow{(1)} & \text{Enhanced noise} & \xrightarrow{(2)} & \text{(PAM)} & \xrightarrow{(3)} & \text{Generator} \\ \xi & & Q(\xi) & & u & & \mathcal{H} \end{matrix}$$

Step (1) is standard in the field of singular stochastic PDEs: it associates in a measurable (but typically non-continuous) way an enhanced noise to a (typically rough) noise. The point is that the additional data contained in the enhanced noise will allow to recover continuity in the subsequent steps. Note that the renormalisation is implemented in this step.

Step (2) is deterministic: given the enhanced noise $Q(\xi)$, one defines a solution theory to the (PAM). In dimension 2 it relies on relatively elementary arguments that we recall in section III.3. On the other hand, the construction of (PAM) in dimension 3 is involved and relies on the theory of regularity structures [Hai14]: we will use the results from [HL18], see section III.4.

If we denote by $u^f(t,\cdot)$ the solution of (PAM), then $P_t f := u^f(t,\cdot)$ is a semigroup. Step (3) builds (in a deterministic way) a unique self-adjoint operator $\mathcal{H}$ satisfying $P_t f = e^{-t\mathcal{H}} f$. However, extracting $\mathcal{H}$ from this semigroup is much more subtle than one may think.
A natural attempt would be to construct the resolvent of $\mathcal{H}$ by integration in time of the semigroup: $\int_0^\infty e^{-at} u^f(t,\cdot)dt$. However the operator is not bounded from below so that

nothing ensures that $(\mathcal{H}+a)^{-1}f$ exists for real parameters a. In addition, the solution theory of (PAM) provides very bad a priori bounds on the growth in time of $u^f(t,\cdot)$ so that one cannot single out nice initial conditions f for which the above integral in time converges. Actually these functions f for which the integral in time converges should be functions that, informally speaking, "avoid" these regions of space where ξ is very large, thus confirming that one cannot identify a simple set of such functions.
A second approach would consist in applying Friedrich's extension to the quadratic forms $\langle f, u^f(t,\cdot)\rangle$ (for nice enough functions f) and in exploiting the semigroup property satisfied by the solution of (PAM): we did not manage to conclude with this approach. We then came accross a (beautiful) result of Klein and Landau [KL81] that provides the right framework. This is presented in section III.2, and the required estimates on the (PAM) to apply this result and deduce Theorem III.1 are presented in sections III.3 and III.4.

Let us mention that the construction of the operator via its semigroup comes with nice continuity properties, that allow, in particular, to prove the strong resolvent convergence of the statement, see Propositions III.12 and III.28. We would like to emphasise that the approach presented here is robust and can certainly be applied to a large class of singular random differential operators.

Our second result is the identification of the spectrum of $\mathcal{H}$.

Theorem III.2. *In dimensions* 2 *and* 3*, almost surely the spectrum of* $\mathcal{H}$ *is* $\mathbb{R}$*.*

The fact that the spectrum is almost surely a deterministic set is a consequence of the ergodicity of white noise combined with the following commutation property involving the shifted noise $\theta_x\xi(\cdot)=\xi(\cdot-x)$ and the translation operators $\mathcal{T}_x f(\cdot)=f(\cdot+x)$ (see section III.5 for more details)

$$\mathcal{H}(\theta_x\xi)=\mathcal{T}_x^*\mathcal{H}(\xi)\mathcal{T}_x\ ,\quad \forall x\in\mathbb{R}^d\ .$$

While this property is clear at an informal level, to establish it rigorously one needs to construct the enhanced noise associated to the shifted noise $\theta_x\xi$ simultaneously for all $x\in\mathbb{R}^d$. This is the purpose of Lemmas III.4 and III.21.

A usual strategy to show that some value $\lambda\in\mathbb{R}$ belongs to the spectrum of a self-adjoint operator A is to build a Weyl sequence, namely, a sequence of functions f_n of unit L^2-norm which are such that $(A-\lambda)f_n$ converges to 0 in L^2. This argument is implemented for the operator $\mathcal{H}_\varepsilon$ in Proposition III.31, and allows to show that the spectrum of the latter is the whole line $\mathbb{R}$ a.s. For the limiting operator $\mathcal{H}$ however, this argument is delicate to implement since the domain of the operator is made of non-smooth functions that depend in a subtle way on the enhanced noise. Let us mention that Ueki [Uek23] managed to implement Weyl sequences in dimension 2 using delicate approximation arguments, and then showed that the spectrum is the whole line $\mathbb{R}$ almost surely.

In the present work, we rely on a different argument that works both in dimensions 2 and 3. The underlying idea comes from a result of Kotani [Kot85] which can be spelled out in the following way: if one shows that the (topological) support of ξ_ε is included into the (topological) support of ξ, then the spectrum of $\mathcal{H}_\varepsilon$ is included in the spectrum of $\mathcal{H}$. Since Proposition III.31 shows that the former is $\mathbb{R}$, we deduce the asserted result. While this informal idea is rigourous for function-valued potentials as shown in [Kot85], in the singular context considered in this article it is only heuristic. Indeed, the argument of Kotani relies on the continuity of the operator w.r.t. to the noise: in the present situation, the operator is only measurable w.r.t. the noise but is continuous w.r.t. the enhanced noise. Consequently, we need to show that the (topological) support of the *enhanced noise* associated to ξ_ε is included into the (topological) support of the enhanced noise associated to ξ, and we will show in section III.5 that we can extend the argument of Kotani to that setting.
In dimension 2, we provide a complete proof of the inclusion of the supports of the enhanced noises, see Subsection III.3.4. In dimension 3, we rely on a general result of Hairer and Schönbauer [HS21] that establishes a support theorem for singular stochastic PDE, see Subsection III.4.4.

Notations

Let us introduce some notations for the rest of the article. For some given parameters $a > 0$ and $\ell \in \mathbb{R}$, we introduce the weight functions

$$p_a(x) = (1 + |x|)^a \;, \quad e_\ell(x) = e^{\ell(1+|x|)} \;, \quad x \in \mathbb{R}^d \;. \tag{III.2}$$

Let us mention that for any given $a > 0$, there exists a constant $C > 0$ such that for all $0 \le s \le t$ and all $\ell \in \mathbb{R}$

$$\sup_{x\in\mathbb{R}^d} \frac{p_a(x)e_{\ell+s}(x)}{e_{\ell+t}(x)} \le C(t-s)^{-a} \;. \tag{III.3}$$

For $w = p_a$ or $w = e_\ell$, we let L^2_w be the weighted version of the usual L^2 space: its norm is defined through

$$\|f\|^2_{L^2_w} := \int_{x\in\mathbb{R}^d} \frac{|f(x)|^2}{w(x)} dx \;.$$

For $\alpha < 0$, we let $\mathcal{C}^\alpha_w$ be the closure of all compactly supported, smooth functions under the norm

$$\|f\|_{\mathcal{C}^\alpha_w} := \sup_{x\in\mathbb{R}^d} \sup_{\lambda\in(0,1]} \sup_{\varphi\in\mathcal{B}_{\lceil-\alpha\rceil}} \frac{|\langle f, \varphi^\lambda_x\rangle|}{w(x)} \;,$$

where $\varphi^\lambda_x(y) := \lambda^{-d}\varphi((y-x)/\lambda)$ and $\mathcal{B}_r$ is the set of all functions, supported in the unit ball of $\mathbb{R}^d$ whose C^r-norm is bounded by 1. Similarly we let $\mathcal{H}^\alpha_w$ be the weighted Besov space with parameters $p = q = 2$ (or equivalently, the weighted Sobolev space) of regularity index α.
We point out a fact that will be used many times in this article. Let $\chi : \mathbb{R}^d \to [0,1]$

be a compactly supported smooth function such that $\sum_{k\in\mathbb{Z}^d}\chi(\cdot-k)=1$. Then for any given weight w, there exists a constant $C>0$

$$\|f\|_{\mathcal{C}^\alpha_w}\le C\sup_{k\in\mathbb{Z}^d}\frac{\|f(\cdot)\chi(\cdot-k)\|_{\mathcal{C}^\alpha}}{w(k)}\ . \tag{III.4}$$

III.2 SELF-ADJOINT GENERATOR OF UNBOUNDED SYMMETRIC SEMIGROUP

This section presents a general result due to Klein and Landau. To ease the reading, let us anticipate its application to the context of (PAM): we will take $P_t f$ as the solution $u(t,\cdot)$ of the PDE (III.1) starting from f and the domain $\mathcal{D}_t$ will consist of a deterministic set of functions $f\in L^2(\mathbb{R}^d)$ which decay sufficiently fast at infinity; let us emphasise that the decay rate will depend on t so that the sets $\mathcal{D}_t$ will decrease with time. In this context, it seems quite difficult to extract the resolvents from the semigroup. Instead, Klein and Landau construct the spectral measures to establish the following theorem.

Theorem III.3 ([KL81])**.** *Let $(P_t)_{t\in[0,T]}$, $T>0$, be a collection of unbounded operators on a Hilbert space $\mathfrak{H}$ such that $P_0=I$ and P_t has domain $\mathcal{D}_t$. Assume $(P_t)_{t\in[0,T]}$ satisfies the following properties:*

1. *(Non-increasing domains with dense union) For $s\le t$, $\mathcal{D}_t\subset\mathcal{D}_s$ and $\bigcup_{0<t\le T}\mathcal{D}_t$ is dense in $\mathfrak{H}$.*
2. *(Semigroup) For $0\le s\le t\le T$, $P_s\mathcal{D}_t\subset\mathcal{D}_{t-s}$ and $P_{t-s}P_s=P_t$ on $\mathcal{D}_t$.*
3. *(Symmetry) For $f,g\in\mathcal{D}_t$, $\langle f,P_t g\rangle_{\mathfrak{H}}=\langle P_t f,g\rangle_{\mathfrak{H}}$.*
4. *(Weak continuity) For $t\in(0,T]$ and $f\in\mathcal{D}_t$, the function $s\mapsto\langle f,P_s f\rangle_{\mathfrak{H}}$ is continuous on the interval $[0,t]$.*

Then, there exists a unique self-adjoint operator H on $\mathfrak{H}$ such that for every $t\in[0,T]$, $\mathcal{D}_t\subset\mathcal{D}(e^{-tH})$ and such that P_t is the restriction of e^{-tH} to $\mathcal{D}_t$. Moreover, the operator H is essentially self-adjoint on the core $\hat{\mathcal{D}}:=\bigcup_{0<t\le S}\bigcup_{0<s<t}P_s\mathcal{D}_t$, for any $S\in(0,T]$.

Let us summarize the main steps of the proof. The key observation is the following: for any given $f\in\mathcal{D}_t, t\in(0,T]$, the assumptions of the theorem ensure that the map $r:s\mapsto\left\|P_{s/2}f\right\|^2_{\mathfrak{H}}$ is continuous, non-negative on $[0,t]$ and satisfies what the authors of [KL81] call *Osterwalder-Schrader positivity*, or *reflection positivity* in the mathematical physics community: for all $(t_i)_{1\le i\le n}\in[0,T]$ and all $(c_i)_{1\le i\le n}\in\mathbb{R}$

$$\sum_{j,k=1}^n c_j c_k r(t_j+t_k)\ge 0\ .$$

A theorem of Widder [Wid34], which can be thought of as a real version of the theorem of Bochner on characteristic functions of probability measures, implies that the map r

can be represented as the Laplace transform of a unique positive measure ν on $\mathbb{R}$, i.e. $\left\|P_{s/2}f\right\|_{\mathfrak{H}}^2 = \int_{\mathbb{R}} e^{-sa}\nu(\mathrm{d}a)$. Let us point out that ν turns out to be the spectral measure (associated to f) of the self-adjoint operator H to be defined.

Since the Laplace transform of ν exists for $s \in [0,t]$, we can consider $\int_{\mathbb{R}} e^{-za}\nu(\mathrm{d}a)$ for all $z \in \mathbb{C}$ with $\mathrm{Re}\, z \in [0,t]$. This allows to analytically extend the $\mathfrak{H}$-valued function $s \mapsto P_s f$ into a function F_f defined on $\{z \in \mathbb{C} : \mathrm{Re}\, z \in [0,t]\}$ in such a way that $F_f(s) = P_s f$ for $s \in [0,t]$ and that $\langle F_f(z_1), F_f(z_2)\rangle_{\mathfrak{H}} = \int_{\mathbb{R}} e^{-(\bar{z_1}+z_2)a}\nu(\mathrm{d}a)$ for any $z_1, z_2 \in \mathbb{C}$ with $\mathrm{Re}(z_j) \in [0,T]$. As one can show that $F_f(z)$ is linear in f, it makes sense to set $U(y)f = F_f(iy)$ for all $f \in \mathcal{D}_t, y \in \mathbb{R}$ and one has $\|U(y)f\|_{\mathfrak{H}}^2 = \|F_f(0)\|_{\mathfrak{H}}^2 = \|f\|_{\mathfrak{H}}^2$. So far the construction works simultaneously for all $t \in (0,T]$, $U(y)f$ is therefore well-defined for all f in the dense subspace $\bigcup_{0<t\le T} \mathcal{D}_t$. We can thus continuously extend $U(y)$ into a unitary operator on $\mathfrak{H}$. One can then show that $(U(y))_{y\in\mathbb{R}}$ forms a strongly continuous one-parameter group of unitary operators. An application of Stone's Theorem then yields a unique self-adjoint operator H such that $e^{-iyH} = U(y)$. By continuity $e^{-sH}f = P_s f$ for all $s \in [0,t]$.

III.3 THE ANDERSON HAMILTONIAN IN DIMENSION TWO

The goal of this section is to construct the Anderson Hamiltonian with white noise potential on $\mathbb{R}^2$. To that end, we will feed Theorem III.3 with the solution to the parabolic Anderson model

$$\begin{cases} \partial_t u = \Delta u - u\xi \;, & \text{on } \mathbb{R}^2 \;, \\ u(t=0,\cdot) = f(\cdot) \;. \end{cases} \tag{III.5}$$

An analysis of the regularities at stake shows that this PDE is ill-defined, and, actually requires renormalization by infinite constants. Let us explain how it can be performed. Recall the smoothed out noise ξ_ε of the introduction. Denote by u_ε the solution of the above PDE in which $-u\xi$ is replaced by $-u_\varepsilon(\xi_\varepsilon - C_\varepsilon)$ for some well-chosen constant C_ε. Set $G = -\frac{\log|x|}{2\pi}\chi(x)$ where χ is a smooth, radial, cutoff function equal to 1 in the unit ball $B(0,1)$ and supported in $B(0,2)$. The function G coincides with the Green's function of the Laplacian in the unit ball and one can check that $-\Delta G = \delta_0 + F$ for some smooth function F supported in $B(0,2)\backslash B(0,1)$.
We rely on the change of unknown $w_\varepsilon := e^{Y_\varepsilon}u_\varepsilon$ where $Y_\varepsilon := G * \xi_\varepsilon$. It turns out that w_ε solves the following PDE

$$\begin{cases} \partial_t w_\varepsilon = \Delta w_\varepsilon - 2\nabla Y_\varepsilon \cdot \nabla w_\varepsilon + (|\nabla Y_\varepsilon|^2 - C_\varepsilon + F * \xi_\varepsilon)w_\varepsilon \;, & t > 0 \;, x \in \mathbb{R}^2 \;, \\ w_\varepsilon(t=0,\cdot) = e^{Y_\varepsilon} f \;. \end{cases} \tag{III.6}$$

The main observation is that there exists an appropriate choice of C_ε for which $|\nabla Y_\varepsilon|^2 - C_\varepsilon$ converges in probability as $\varepsilon \downarrow 0$ to some well-defined random distribution. In addition, the regularities at stake are such that the limit as $\varepsilon \downarrow 0$ of this PDE is well-posed: the change of unknown "improved" the regularity of the random terms on the r.h.s. The enhanced noise in this specific case must include the limit in probability of $|\nabla Y_\varepsilon|^2 - C_\varepsilon$.

In Subsection III.3.1 we introduce the space of enhanced noise and perform the renormalisation: it corresponds to Step 1 of the introduction. In Subsection III.3.2 we construct the solution theory to a PDE corresponding to (III.6): this implements Step 2 of the introduction. In Subsection III.3.3 we define the semigroup and its generator: this is Step 3 of the introduction. Finally in Subsection III.3.4 we characterise the support of the enhanced noise. In this whole section, $\kappa > 0$ is a small parameter (it is implicitly taken as small as needed).

III.3.1 The space of enhanced noise and the white noise case

We consider the space of enhanced noises $\mathscr{M} := \mathcal{C}^{-1-\kappa}_{p_\kappa} \times \mathcal{C}^{-\kappa}_{p_\kappa}$ and denote by $q := (X, U)$ a generic element in this space. Endowed with the norm $\|q\| = \|X\|_{\mathcal{C}^{-1-\kappa}_{p_\kappa}} + \|U\|_{\mathcal{C}^{-\kappa}_{p_\kappa}}$, $\mathscr{M}$ is a separable Banach space.

This space of enhanced noises becomes clear on a specific example. Fix some parameter $b \in (0, \kappa/2)$ and set $\Omega := \mathcal{C}^{-1-\kappa}_{p_b}(\mathbb{R}^2)$ endowed with the law $\mathbb{P}$ of white noise. The canonical variable on Ω will be denoted ξ. We define $\xi_\varepsilon := \xi * \varrho_\varepsilon$, $Y_\varepsilon := G * \xi_\varepsilon$ and

$$Z_\varepsilon(x) := |\nabla Y_\varepsilon(x)|^2 - C_\varepsilon \ , \quad C_\varepsilon := \mathbb{E}\left[|\nabla Y_\varepsilon(0)|^2\right] .$$

Note that $x \mapsto Y_\varepsilon(x)$ is a stationary (gaussian) field so that C_ε is left unchanged if we shift the point at which ∇Y_ε is evaluated. We set $Q_\varepsilon := (\xi_\varepsilon, Z_\varepsilon)$, this r.v. takes values in $\mathscr{M}$, see for instance [HL15, Lemma 1.1 and Corollary 1.2]. Let us give some details regarding ξ_ε since it illustrates the role played by the polynomial weight here (and since this argument will appear at several occasions later on): using (III.4) we find for any $n \geq 1$

$$\mathbb{E}\left[\|\xi_\varepsilon\|^n_{\mathcal{C}^{-1-\kappa}_{p_\kappa}}\right] \leq C \sum_{k\in\mathbb{Z}^d} \frac{\mathbb{E}\left[\|\xi_\varepsilon \chi(\cdot - k)\|^n_{\mathcal{C}^\alpha}\right]}{p_\kappa(k)^n} . \tag{III.7}$$

The term $\mathbb{E}\left[\|\xi_\varepsilon \chi(\cdot - k)\|^n_{\mathcal{C}^\alpha}\right]$ is independent of k and is finite for any $n \geq 1$: therefore, choosing n large enough, the previous sum converges.
Note that Z_ε "contains" two instances of ξ so that it requires a weight of order p_{2b}, and since we chose $b < \kappa/2$, it indeed lives in a space with weight p_κ.

For any $x \in \mathbb{R}^2$, we define the shift operator θ_x on Ω as follows: $\theta_x \xi(\cdot) := \xi(\cdot - x)$, or more formally

$$\langle \theta_x \xi, \varphi \rangle = \langle \xi, \varphi(\cdot + x) \rangle \ , \quad \varphi \in C^\infty_c(\mathbb{R}^2) .$$

It is easy to check that the shift operators are continuous maps from Ω into itself. We naturally extend this definition to $\mathscr{M}$ by defining $\theta_x q := (\theta_x X, \theta_x U)$ for any $q = (X, U) \in \mathscr{M}$. The shift operators are continuous maps from $\mathscr{M}$ into itself.

To identify the spectrum of $\mathcal{H}$ in section III.5, we will need to show a commutation property: the operator $\mathcal{H}(\theta_x \xi)$ associated to the shifted noise coincides with the operator

$\mathcal{H}(\xi)$ conjugated with spatial shifts, see Lemma III.30. Therefore we need to argue that one can deal simultaneously with all the shifted noises: this is the purpose of the following technical lemma.

Lemma III.4. *There exists a sequence $\varepsilon_k \downarrow 0$ such that the set*

$$\Omega_0 := \left\{ \xi \in \Omega : Q_{\varepsilon_k}(\theta_x \xi) \text{ converges in } \mathscr{M} \text{ as } k \to \infty\ , \quad \forall x \in \mathbb{R}^2 \right\} , \tag{III.8}$$

is of full $\mathbb{P}$-measure and is invariant under all $\theta_x, x \in \mathbb{R}^2$. The[1] *limit Q satisfies $Q(\theta_x \xi) = \theta_x Q(\xi)$ for all $\xi \in \Omega_0$.*
As $\varepsilon \downarrow 0$, the field $(Q_\varepsilon(\theta_x\xi))_{x\in\mathbb{R}^2}$ which takes values in $\mathscr{M}$ converges in probability, locally uniformly over $x \in \mathbb{R}^2$, to $(Q(\theta_x\xi))_{x\in\mathbb{R}^2}$.

Proof. We start with a general bound. Let $f \in \mathcal{C}^\beta_{p_\kappa}$ for some $\beta < 0$. Then for any $x \in \mathbb{R}^d$

$$\|\theta_x f\|_{\mathcal{C}^\beta_{p_\kappa}} \le \sup_{y\in\mathbb{R}^d} \frac{p_\kappa(y-x)}{p_\kappa(y)} \|f\|_{\mathcal{C}^\beta_{p_\kappa}} = p_\kappa(x) \|f\|_{\mathcal{C}^\beta_{p_\kappa}} .$$

This inequality is a direct consequence of the definitions of the shift and of the $\mathcal{C}^\beta_{p_\kappa}$-norm. It implies that

$$\sup_{x\in\mathbb{R}^2} \frac{1}{p_\kappa(x)} \|\theta_x f\|_{\mathcal{C}^\beta_{p_\kappa}} \le \|f\|_{\mathcal{C}^\beta_{p_\kappa}} .$$

It is straightforward to check that for any $\varepsilon \in (0,1)$, any $x \in \mathbb{R}^2$ and any $\xi \in \Omega$

$$Q_\varepsilon(\theta_x \xi) = \theta_x Q_\varepsilon(\xi) .$$

This observation combined with the inequality proven above implies that for all $\xi \in \Omega$ and all $\varepsilon, \varepsilon' \in (0,1)$

$$\sup_{x\in\mathbb{R}^2} \frac{1}{p_\kappa(x)} \|Q_\varepsilon(\theta_x\xi) - Q_{\varepsilon'}(\theta_x\xi)\|_{\mathscr{M}} \le \|Q_\varepsilon(\xi) - Q_{\varepsilon'}(\xi)\|_{\mathscr{M}} .$$

The arguments in [HL15, Lemma 1.1, Proposition 1.3] show that, for any given $p > 1$, $Q_\varepsilon(\xi)$ converges in $L^p(\Omega, \mathbb{P})$ as $\varepsilon \downarrow 0$ to some limit $Q(\xi) = (\xi, Z)$ in $\mathscr{M}$. We thus deduce that the collection of r.v. $(Q_\varepsilon(\theta_x\xi), x \in \mathbb{R}^2)$, taking values in $L^\infty_{p_\kappa}(\mathbb{R}^2 \to \mathscr{M})$, converges as $\varepsilon \downarrow 0$ in $L^p(\Omega, \mathbb{P})$. Therefore there exists a deterministic sequence $\varepsilon_k \downarrow 0$ such that the convergence holds almost surely. We thus consider the set of full $\mathbb{P}$-measure

$$\Omega_0 := \left\{ \xi \in \Omega : Q_{\varepsilon_k}(\theta_x \xi) \text{ converges in } \mathscr{M} \text{ as } k \to \infty\ , \quad \forall x \in \mathbb{R}^2 \right\} .$$

It is elementary to check that Ω_0 is invariant under all shifts θ_x, $x \in \mathbb{R}^2$. Let us denote by $\tilde{Q}(x,\xi) := \lim_k Q_{\varepsilon_k}(\theta_x\xi)$ for all $x \in \mathbb{R}^2$ and all $\xi \in \Omega_0$. For any $\xi \in \Omega_0$ and any $x \in \mathbb{R}^2$

$$\tilde{Q}(x,\xi) = \lim_k Q_{\varepsilon_k}(\theta_x\xi) = \lim_k Q_{\varepsilon_k}(\theta_0(\theta_x\xi)) = \tilde{Q}(0, \theta_x\xi) ,$$

[1] Q is defined as a limit on the set of full measure Ω_0, and is extended to Ω arbitrarily.

and, using the continuity of the shift operator on $\mathscr{M}$

$$\tilde{Q}(x,\xi) = \lim_k Q_{\varepsilon_k}(\theta_x \xi) = \lim_k \theta_x Q_{\varepsilon_k}(\xi) = \theta_x \tilde{Q}(0,\xi) \ .$$

If we now set $Q(\xi) := \tilde{Q}(0,\xi)$, we have shown that for all $\xi \in \Omega_0$

$$Q(\theta_x \xi) = \theta_x Q(\xi) \ .$$

To conclude the proof, let us assign arbitrary values to $Q(\xi)$ for $\xi \in \Omega \backslash \Omega_0$. We recall that $(Q_\varepsilon(\theta_x\xi), x \in \mathbb{R}^2)$ converges in $L^p(\Omega, \mathbb{P})$. Necessarily its limit coincides with the a.s. limit along ε_k, that is $(Q(\theta_x \xi), x \in \mathbb{R}^2)$. □

III.3.2 The parabolic evolution

For a given $q = (X,U) \in \mathscr{M}$, we set $V := G * X$. One can check that V belongs to $\mathcal{C}^{1-\kappa}_{p_\kappa}$. Given the *enhanced noise* $q = (X,U)$, we consider the PDE

$$\begin{cases} \partial_t w = \Delta w - 2\nabla V \cdot \nabla w + (U + F * X) w \ , \quad t > 0 \ , x \in \mathbb{R}^2 \ , \\ w(t=0,\cdot) = w_0 \ . \end{cases} \tag{III.9}$$

Remark III.5. If one takes $X = \xi_\varepsilon$ and $U = |\nabla V|^2 - C_\varepsilon$, then w coincides with w_ε of (III.6).

Theorem III.6. *The mild solution to* (III.9) *yields a map* $(q,f,t) \mapsto w^{q,f}(t,\cdot)$ *defined on* $\mathscr{M} \times \left(\bigcup_{\ell_0 \in \mathbb{R}} L^2_{e_{\ell_0}} \right) \times \mathbb{R}_+$ *such that for any* $\ell_0 \in \mathbb{R}$*:*

1. *(Continuity in the data) For any* $t \geq 0$*, the map* $(q,f) \mapsto w^{q,f}(t,\cdot)$ *is continuous from* $\mathscr{M} \times L^2_{e_{\ell_0}}$ *into* $L^2_{e_{\ell_0+t}}$*, and is linear w.r.t.* f*.*

2. *(Continuity in time) For any* $(q,f) \in \mathscr{M} \times L^2_{e_{\ell_0}}$ *and any* $T > 0$*, the map* $t \mapsto w^{q,f}(t,\cdot)$ *is continuous from* $[0,T]$ *into* $L^2_{e_{\ell_0+T}}$*.*

3. *(Semigroup property) For any* $(q,f) \in \mathscr{M} \times L^2_{e_{\ell_0}}$ *and any* $0 \leq s \leq t$*,* $w^{q,f}(t,\cdot)$ *coincides with* $w^{q,g}(s,\cdot)$ *where* $g = w^{q,f}(t-s,\cdot)$*.*

Proof. This is essentially an adaptation of [HL15], except for property 2. Let us present the main arguments. Given a time horizon $T > 0$, and some parameters $\ell \in \mathbb{R}$, $\alpha, \beta \geq 0$ we define $\mathcal{E}_{\ell,T,\alpha,\beta}$ as the completion of all smooth functions $u : (0,T] \times \mathbb{R}^2 \to \mathbb{R}$ under the norm

$$|\!|\!| u |\!|\!|_{\ell,T,\alpha,\beta} := \sup_{t \in (0,T]} t^\beta \left\| u(t,\cdot) \right\|_{\mathcal{H}^\alpha_{e_{\ell+t}}} < \infty \ . \tag{III.10}$$

Set $g = U + F * X$. Since F is smooth, we have $g \in \mathcal{C}^{-\kappa}_{p_\kappa}$. For simplicity we write $w(t) = w(t,\cdot)$ and we set

$$\mathcal{M}_{T,f} w(t) := K_t * f + \int_0^t K_{t-s} * (-2\nabla V \cdot \nabla w(s) + g w(s))\, \mathrm{d}s, \quad t \in (0,T] \tag{III.11}$$

with K_t denoting the heat kernel. This corresponds to the mild formulation of (III.9). We are going to show that $\mathcal{M}_{T,f}$ admits a unique fixed point in $\mathcal{E}_{\ell_0,T,\alpha,\beta}$ provided T is small enough.
Set $\alpha = 1 + 2\kappa$ and $\beta = \frac{\alpha}{2} = \frac{1}{2} + \kappa$. By the Schauder estimate for the heat kernel and the bound (III.3), one has

$$\|K_t * f\|_{\mathcal{H}^{\alpha}_{e_{\ell_0}}} \lesssim t^{-\beta} \|f\|_{L^2_{e_{\ell_0}}} ,$$
$$\|K_{t-s} * (-2\nabla V \cdot \nabla w(s) + gw(s))\|_{\mathcal{H}^{\alpha}_{e_{\ell_0+t}}} \lesssim (t-s)^{-\frac{\alpha+\kappa}{2}-\kappa} s^{-\beta} \|q\|_{\mathscr{M}} \|w\|_{\ell_0,T,\alpha,\beta} .$$

Since κ can be chosen arbitrarily small, one can make $\frac{\alpha}{2} + \frac{3\kappa}{2} < 1$ and $\beta < 1$. The integral over time in (III.11) therefore converges and is of order $\int_0^t (t-s)^{-\frac{\alpha+\kappa}{2}-\kappa} s^{-\beta}\, \mathrm{d}s \lesssim t^{1-\frac{\alpha}{2}-\frac{3}{2}\kappa} t^{-\beta}$, whence

$$\|\mathcal{M}_{T,f} w\|_{\ell_0,T,\alpha,\beta} \lesssim \|f\|_{L^2_{e_{\ell_0}}} + T^{1-\frac{\alpha}{2}-\frac{3}{2}\kappa} \|q\|_{\mathscr{M}} \|w\|_{\ell_0,T,\alpha,\beta}. \tag{III.12}$$

Therefore, $\mathcal{M}_{T,f}$ is a well-defined map on $\mathcal{E}_{\ell_0,T,\alpha,\beta}$. In the same vein, for any two elements $w, \bar{w} \in \mathcal{E}_{\ell_0,T,\alpha,\beta}$, the previous calculation shows

$$\|\mathcal{M}_{T,f}(w - \bar{w})\|_{\ell_0,T,\alpha,\beta} \lesssim T^{1-\frac{\alpha}{2}-\frac{3}{2}\kappa} \|q\|_{\mathscr{M}} \|w - \bar{w}\|_{\ell_0,T,\alpha,\beta}.$$

Since $1 - \frac{\alpha}{2} - \frac{3}{2}\kappa > 0$, $\mathcal{M}_{T,w_0}$ can be made contracting by choosing a T small enough (depending only on q), in which case there exists a unique fixed point in $\mathcal{E}_{\ell_0,T,\alpha,\beta}$ that we denote $w^{q,f}$. As T depends only on q, this solution map can be extended to global in time solution by an iterative argument consisting in restarting the equation from the initial data $w^{q,f}(T)$, $w^{q,f}(2T)$, etc. We have now proved the existence and uniqueness of the solution map $(q, f, t) \mapsto w^{q,f}(t)$.

The semigroup property stated as point 3. follows from the semigroup property of the heat kernel and from the equation satisfied by the unique fixed point of the map defined above. The linearity in f is immediate from (III.11). Now we prove the continuity of $(q, f) \mapsto w^{q,f}$. Fix any $R > 0$ and take two elements $(q_j = (X_j, U_j), f_j)$, $j = 1, 2$ in the ball of radius R of $\mathscr{M} \times L^2_{e_{\ell_0}}$. Denote $V_j = G * X_j$ and $g_j = U_j + F * X_j$, one has

$$\begin{aligned} w^{q_1,f_1}(t) - w^{q_2,f_2}(t) = K_t * (f_1 - f_2) + \int_0^t K_{t-s} * [&-2\nabla(V_1 - V_2) \cdot \nabla w^{q_1,f_1}(s) \\ &- 2\nabla V_2 \cdot \nabla(w^{q_1,f_1} - w^{q_2,f_2}) + (g_1 - g_2) w^{q_1,f_1}(s) + g_2(w^{q_1,f_1} - w^{q_2,f_2})(s)]\, \mathrm{d}s. \end{aligned}$$

A calculation similar to the one above shows

$$\begin{aligned} &\left\| w^{q_1,f_1} - w^{q_2,f_2} \right\|_{\ell_0,T,\alpha,\beta} \lesssim \|f_1 - f_2\|_{L^2_{e_{\ell_0}}} \\ &+ T^{1-\frac{\alpha}{2}-\frac{3}{2}\kappa} \left(\|q_1 - q_2\|_{\mathscr{M}} \left\| w^{q_1,f_1} \right\|_{\ell_0,T,\alpha,\beta} + R \left\| w^{q_1,f_1} - w^{q_2,f_2} \right\|_{\ell_0,T,\alpha,\beta} \right). \end{aligned}$$

By choosing T small enough (depending only on R), we then deduce $(q, f) \mapsto w^{q,f}$ is uniformly continuous on the ball with radius R in $\mathscr{M} \times L^2_{e_{\ell_0}}$. Since $R > 0$ is arbitrary, we have thus proved the continuity for fixed t small enough. Again by an iterative argument, this can be extended to any $t > 0$.

We turn to point 2., the strong continuity in time of $w = w^{q,f}$. First we observe that the heat operator K_t is a bounded operator on $L^2_{e_\ell}$ for any given $\ell \in \mathbb{R}$, and that it is strongly continuous in t.
We now prove the right continuity. Fix $t \in [0, T)$. For $\varepsilon > 0$ such that $t + \varepsilon \leq T$, we compute

$$\|w(t+\epsilon) - w(t)\|_{L^2_{e_{\ell_0+t+\varepsilon}}} \leq \|K_\varepsilon w(t) - w(t)\|_{L^2_{e_{\ell_0+t+\varepsilon}}} + \left\| \int_0^\epsilon K_{\varepsilon - s} * (-2\nabla V \cdot \nabla w(t+s) + g w(t+s))\, \mathrm{d}s \right\|_{L^2_{e_{\ell_0+t+\varepsilon}}}$$

The second term is controlled by

$$\int_0^\varepsilon (\varepsilon - s)^{-\frac{3}{2}\kappa}(t+s)^{-\beta} \|q\|_{\mathscr{M}} \|\!|w|\!\|_{\ell_0,T,\alpha,\beta}\, \mathrm{d}s \lesssim \varepsilon^{1-\frac{3}{2}\kappa}.$$

The right continuity thus follows from the fact that $1 - \frac{3}{2}\kappa > 0$ and the strong continuity of the heat kernel.
For the left continuity, fix $t \in (0, T]$ and write $w(t) - w(t-\varepsilon) = I_1 + I_2 + I_3$ where

$$I_1 := (K_t - K_{t-\varepsilon})f\ , \quad I_2 := \int_0^{t-\varepsilon} (K_{t-s} - K_{t-\varepsilon-s})(-2\nabla V \cdot \nabla w(s) + g w(s))\, \mathrm{d}s\ ,$$
$$I_3 := \int_{t-\varepsilon}^t K_{t-s}(-2\nabla V \cdot \nabla w(s) + g w(s))\, \mathrm{d}s\ .$$

Obviously the $L^2_{e_{\ell_0+t}}$-norm of I_1 goes to 0 as $\varepsilon \to 0$ by the strong continuity of S_ε on $L^2_{e_{\ell_0+t}}$. For I_3, we have a similar control as previously:

$$\|I_3\|_{L^2_{e_{\ell_0+t}}} \lesssim \int_{t-\varepsilon}^t (t-s)^{-\frac{3}{2}\kappa} s^{-\beta} \|q\| \|\!|w|\!\|_{\ell_0,T,\alpha,\beta}\, \mathrm{d}s \lesssim \varepsilon^{1-\frac{3}{2}\kappa}(t-\varepsilon)^{-\beta}.$$

For I_2, we use $K_{t-s} - K_{t-s-\varepsilon} = K_{(t-s)/2-\varepsilon}(K_\varepsilon - I)K_{(t-s)/2}$ and that $\left\|K_{(t-s)/2-\varepsilon}\right\| \leq 1$ as a bounded operator on $L^2_{e_\ell}$ for any given $\ell \in \mathbb{R}$. It gives

$$\|I_2\|_{L^2_{e_{\ell_0+t}}} \leq \int_0^{t-\varepsilon} h_\varepsilon(s)\, \mathrm{d}s$$

where

$$h_\varepsilon(s) = \left\|(K_\varepsilon - I)K_{\frac{t-s}{2}}(-2\nabla V \cdot w(s) + g w(s))\right\|_{L^2_{e_{\ell_0+t}}}, \quad s \in [0, t].$$

Note that by the strong continuity of K_ε, one has $h_\varepsilon(s) \to 0$ for all $s \in [0, t]$; moreover, uniformly for ε we have $h_\varepsilon(s) \lesssim (t-s)^{-\frac{3}{2}\kappa} s^{-\beta} \|q\|_{\mathscr{M}} \|\!|w|\!\|_{\ell_0,T,\alpha,\beta}$ which is an L^1-function on $[0, t]$. By dominated convergence it follows that $\|I_2\|_{L^2_{e_{\ell_0+t}}} \leq \int_0^t h_\varepsilon(s)\, \mathrm{d}s \to 0$ as $\varepsilon \to 0$.
Since I_1, I_2 and I_3 all converge to 0 as $\varepsilon \to 0$, we have thus proved the left-continuity of w. □

III.3.3 The semigroup

Definition III.7. Fix $\delta > 0$ and $q \in \mathscr{M}$. For any $t > 0$, define the domain $\mathcal{D}_t = \mathcal{D}_t(q) := e^{-V} L^2_{e_{-t-\delta}} \subset L^2$ and the operator

$$P_t : \begin{cases} \mathcal{D}_t & \to \quad L^2 \\ f & \mapsto \quad e^{-V} w^{q,e^V f}(t) \end{cases}$$

Define also $P_0 = I$ on L^2.

Remark III.8. If one sets $u = e^{-V} w^{q,e^V f}$ then u formally solves

$$\begin{cases} \partial_t u = \Delta u + (U - X - |\nabla V|^2)u, \quad t > 0,\ x \in \mathbb{R}^2, \\ u(t=0,\cdot) = f\ . \end{cases} \tag{III.13}$$

This PDE makes sense when X, U are functions, but is only formal in general as the terms $|\nabla V|^2$, Xu are singular products.

In the sequel, we regard P_t as an unbounded operator on the Hilbert space $\mathfrak{H} = L^2(\mathbb{R}^2, dx)$ with domain $\mathcal{D}_t$. The following proposition shows that for any given $T > 0$, the collection $(P_t)_{t\in[0,T]}$ satisfies the assumptions of Theorem III.3.

Proposition III.9. *Fix $T > 0$. The collection $(P_t)_{t\in[0,T]}$ is a semigroup of symmetric operators on $\mathfrak{H} = L^2(\mathbb{R}^2, dx)$ and is strongly continuous with respect to t. In particular, it satisfies the* (Non-increasing domains with dense union)*,* (Semigroup)*,* (Symmetry) *and* (Weak continuity) *properties of Theorem III.3.*

Proof. Non-increasing domains with dense union: For $0 \le s \le t \le T$, we have obviously $\mathcal{D}_t = e^{-V} L^2_{e_{-t-\delta}} \subset e^{-V} L^2_{e_{-s-\delta}} = \mathcal{D}_s$. Moreover, the set $e^{-V} C_c^\infty$ is contained in $L^2_{e_{-t-\delta}}$, therefore $C_c^\infty \subset \mathcal{D}_t$ for all $t \in (0, T]$ so that $\mathcal{D}_t$ is dense in L^2.
Semigroup: Let $f \in \mathcal{D}_t$. The first property of Theorem III.6 shows that $g := w^{q,e^V f}(s) \in L^2_{e_{-(t-s)-\delta}}$ and therefore $P_s f = e^{-V} g \in \mathcal{D}_{t-s}$. In addition the third property of Theorem III.6 shows that

$$P_t f = e^{-V} w^{q,e^V f}(t) = e^{-V} w^{q,g}(t-s)\ ,$$

that is $P_t f = P_{t-s}(e^{-V} g) = P_{t-s}(P_s f)$ as required.
Strong and weak continuity: This is a direct consequence of the second property of Theorem III.6.
Symmetry: By the continuity properties stated in Theorem III.6, it suffices to consider $q = (X, U)$ where X, U are smooth functions with compact support. The symmetry property can be restated at the level of w in the following way

$$\int_{x\in\mathbb{R}^2} w^{q,e^V f}(t,x) e^{-V(x)} g(x) dx = \int_{x\in\mathbb{R}^2} w^{q,e^V g}(t,x) e^{-V(x)} f(x) dx\ .$$

These two terms are the values at $s = 0$ and $s = t$ of the map

$$h(s) := \int_{x\in\mathbb{R}^2} w^{q,e^V f}(s,x) e^{-2V(x)} w^{q,e^V g}(t-s,x) dx\ .$$

Since X, U are smooth, $w^{q,e^V f}$ and $w^{q,e^V g}$ are strong solutions of (III.9) and a direct computation shows that $h(s)$ is constant in s. □

Definition III.10. For $q \in \mathscr{M}$, let $H(q)$ be the unique self-adjoint operator associated, thanks to Theorem III.3, to the symmetric semigroup $(P_t)_{t\in[0,T]}$ of Definition III.7. In particular, P_t is the restriction of $e^{-tH(q)}$ to $\mathcal{D}_t$ for all $t \in [0,T]$.

Remark III.11. Since $\mathcal{D}_t$ is dense in L^2, the proof of [KL81, Lemma 6] shows that $H(q)$ is essentially self-adjoint over $\hat{\mathcal{D}}(q) := \bigcup_{0<s<t} P_s\mathcal{D}_t$. Furthermore, from the proof of Theorem III.6, one can deduce that $\hat{\mathcal{D}}(q) \subset \mathcal{H}^{1-\kappa}(\mathbb{R}^2, dx)$.

We now establish a few properties satisfied by $H(q)$. Given any $f \in L^2$, denote by $\mu_f(q)$ the spectral measure associated to the self-adjoint operator $H(q)$ and f, that is, the unique finite measure on $\mathbb{R}$ with Stieltjes transform

$$\int_{\mathbb{R}} (\lambda - z)^{-1} \mu_f(q)(d\lambda) = \langle f, (H(q) - z)^{-1} f\rangle\ , \quad \forall z \in \mathbb{C}\backslash\mathbb{R}\ .$$

Proposition III.12. *1. For any $f \in L^2$, the map $q \mapsto \mu_f(q)$ is continuous from $\mathscr{M}$ into the space of finite non-negative measures endowed with the topology of weak convergence. As a consequence $q \mapsto H(q)$ is continuous in the strong resolvent sense.*

2. Let $q = (X,U) \in \mathscr{M}$ be such that $X, U \in L^\infty_{p_b}$. Then $H(q)$ coincides with the operator $-\Delta + X + |\nabla G * X|^2 - U$ which is essentially self-adjoint over C_c^∞.

Remark III.13. The point 2 covers the case when $X = \xi_\varepsilon$ and $U = |\nabla G * \xi_\varepsilon|^2 - C_\varepsilon$.

Proof. We start with the first item. Fix $f \in C_c$. Then $f \in \mathcal{D}_t(q)$ for any $q \in \mathscr{M}$ and any $t \geq 0$. In addition, we have

$$\int_{\mathbb{R}} e^{-s\lambda} \mu_f(q)(\mathrm{d}\lambda) = \left\langle f, e^{-sH(q)} f\right\rangle = \left\langle f, e^{-V} w^{q,e^V f}(s)\right\rangle, \quad \forall s \in [0,t]\ .$$

Suppose (q_n) is a sequence in $\mathscr{M}$ converging to q. Since f has compact support, it is easy to check that $e^{V(q_n)} f$ converges to $e^{V(q)} f$ in $L^2_{e_{\ell_0}}$ for any $\ell_0 \in \mathbb{R}$. Then the first property of Theorem III.6 ensures that $w^{q_n,e^{V(q_n)}f}(s)$ converges to $w^{q,e^{V(q)}f}(s)$ in $L^2_{e_{\ell_0+t}}$ for any $\ell_0 \in \mathbb{R}$. Using again the fact that f has compact support, we deduce that

$$\left\langle f, e^{-V(q_n)} w^{q_n,e^{V(q_n)}f}(s)\right\rangle \to \left\langle f, e^{-V(q)} w^{q,e^{V(q)}f}(s)\right\rangle\ , \quad n \to \infty\ .$$

Therefore the Laplace transform of the measure $\mu_f(q_n)$ converges to the Laplace transform of $\mu_f(q)$. By Lemma III.14 below this implies weak convergence of the measures. In turn, it implies that their Stieltjes transforms converge: for any $z \in \mathbb{C}\backslash\mathbb{R}$

$$\int_{\mathbb{R}} (\lambda - z)^{-1} \mu_f(q_n)(\mathrm{d}\lambda) \to \int_{\mathbb{R}} (\lambda - z)^{-1} \mu_f(q)(\mathrm{d}\lambda)\ , \quad n \to \infty\ ,$$

that is $\langle f, (H(q_n) - z)^{-1} f\rangle \to \langle f, (H(q) - z)^{-1} f\rangle$. Since the resolvents at z are uniformly bounded operators and since C_c is dense in L^2, this last convergence remains true for all

$f \in L^2$. In other words, we have shown weak resolvent convergence of $H(q_n)$ to $H(q)$. By [Tes14, Lemma 6.37], this implies strong resolvent convergence, as required.
To conclude the proof of the first item, we show that $\mu_f(q_n)$ converges weakly to $\mu_f(q)$ for any given $f \in L^2$ (so far, we only proved it for $f \in C_c$). The weak resolvent convergence of $H(q_n)$ to $H(q)$ ensures that for any $f \in L^2$, the Stieltjes transform of $\mu_f(q_n)$ converges to the Stieltjes transform of $\mu_f(q)$. Since these measures have a finite mass (equal to the L^2 norm of f), this implies weak convergence.

For the second item, we write $W = X + |\nabla G * X|^2 - U$ which is a locally bounded potential satisfying the bound $W(x) \geq -Cp_\kappa(x)$ for some constant $C > 0$. Since $\kappa < 2$, the operator $T = -\Delta + W$ is essentially self-adjoint over C_c^∞ [Kat72]. Let us still denote by T the self-adjoint extension and let E^T be its projection-valued measure. By functional calculus, for fixed $t > 0$ we can define the unbounded operator e^{-tT}. Fix $f \in \mathcal{D}(e^{-tT})$. By convexity of the function $x \mapsto e^{-\lambda x}$, f lies also in the domain of e^{-sT} for all $s \in [0,t]$. It then makes sense to set $u(s) = e^{-sT} f$ for $s \in [0,t]$, so that u can be regarded as a function from $[0,t]$ to L^2. It can be checked that u solves (III.13) with $q = (X,U)$. Then by the uniqueness stated in Theorem III.6 in a weighted L^2 space, u coincides with $u^{q,f} := e^{-G*X} w^{q,f}$. In particular, one deduces that the L^2-norm of $u(s)$, which is also the Laplace transform of the measure $\left\langle E^T(\cdot)f, f\right\rangle$ evaluated at s, coincides with $\left\|u^{q,f}(s)\right\|_{L^2}^2$. It turns out that the function $s \mapsto \left\|u^{q,f}(s)\right\|_{L^2}^2$ is exactly the Laplace transform of the spectral measure of $H(q)$ associated to the function f and therefore the two Laplace transforms coincide on the interval $s \in [0,t]$. Let E^H denote the projection-valued measure of $H(q)$. It follows that

$$\left\langle E^T(\cdot)f, f\right\rangle = \left\langle E^H(\cdot)f, f\right\rangle, \quad f \in \mathcal{D}(e^{-tT}).$$

By density of $\mathcal{D}(e^{-tT})$, we see that E^T and E^H coincide, which implies $T = H(q)$ by the uniqueness of spectral theorem. □

Lemma III.14. *Let ν_n, ν be finite measures on $\mathbb{R}$. Assume that there exists $T > 0$ such that for all $t \in [0,T]$*

$$\int_{\mathbb{R}} e^{-\lambda t} \nu_n(d\lambda) \to \int_{\mathbb{R}} e^{-\lambda t} \nu(d\lambda)\ , \quad n \to \infty\ ,$$

(implicitly we assume that all these quantities are finite). Then ν_n converges weakly to ν as $n \to \infty$.

Proof. For any $z \in O_T := \{z \in \mathbb{C} : -T < \mathrm{Re}(z) < 0\}$, we set $g_n(z) := \int_{\mathbb{R}} e^{\lambda z} \nu_n(d\lambda)$ and $g(z) := \int_{\mathbb{R}} e^{\lambda z} \nu(d\lambda)$. By assumption $(g_n)_{n\geq 1}$ is a sequence of holomorphic functions on O_T which is uniformly bounded for the supremum norm. By Montel's Theorem, we deduce that there exists a subsequence g_{n_k} that converges uniformly on compact sets of O_T to some holomorphic function h. By uniqueness of analytic continuation, h must coincide with g on O_T and this suffices to deduce that the whole sequence g_n converges to g uniformly on compact sets of O_T. If we set $\mu_n(d\lambda) := e^{-\lambda T/2} \nu_n(d\lambda)$ and $\mu(d\lambda) := e^{-\lambda T/2} \nu_n(d\lambda)$, the previous argument imply that μ_n converges weakly to μ. In turn, this implies that ν_n converges vaguely to ν. Since the total mass of ν_n converges to the total mass of ν, this suffices to conclude. □

We have all the ingredients at hand to define the Anderson Hamiltonian with white noise potential on $\mathbb{R}^2$. For any $\xi \in \Omega$, recall that $Q_\varepsilon(\xi), Q(\xi) \in \mathscr{M}$ and set

$$\mathcal{H}_\varepsilon(\xi) := H(Q_\varepsilon(\xi)) \;, \quad \mathcal{H}(\xi) = H(Q(\xi)) \;,$$

where H is the deterministic map introduced in Definition III.10.

Proof of Theorem III.1 in dimension 2. By Remark III.13, for every $\xi \in \Omega$ we have $\mathcal{H}_\varepsilon(\xi) = -\Delta + \xi_\varepsilon + C_\varepsilon$. By Lemma III.4, Q is the limit in probability of Q_ε so that Proposition III.12 entails that $\mathcal{H}_\varepsilon$ converges in the strong resolvent sense to $\mathcal{H}$ in probability. Finally, Definition III.10 ensures that, for any $t \geq 0$, the domain of the operator $e^{-t\mathcal{H}}$ contains the set $\mathcal{D}_t$ so in particular all functions $f \in L^2(\mathbb{R}^d, dx)$ with compact support, and $e^{-t\mathcal{H}} f$ coincides with the solution of (III.1) at time t, starting from f at time 0. □

III.3.4 Characterization of the support

We conclude this section by proving a crucial result in order to identify the spectrum of $\mathcal{H}$. Let $\mathtt{supp}(Q_\varepsilon)$, resp. $\mathtt{supp}(Q)$, be the topological support of the law of the r.v. Q_ε, resp. Q, that is, the intersection of all closed sets of $\mathscr{M}$ with full measure.

Theorem III.15. *For any $\varepsilon \in (0, 1)$, $\mathtt{supp}(Q_\varepsilon) \subset \mathtt{supp}(Q)$. Furthermore*

$$\mathtt{supp}(Q) = \overline{\{(h, |\nabla G * h|^2 + c) : h \in L^\infty_{p_b}(\mathbb{R}^2) \;,\; c \in \mathbb{R}\}}^{\mathscr{M}} \;.$$

Remark III.16. To identify the spectrum of $\mathcal{H}$, we will only need the inclusion $\mathtt{supp}(Q_\varepsilon) \subset \mathtt{supp}(Q)$. However, our proof of this inclusion relies on the identification of $\mathtt{supp}(Q)$.

The rest of this subsection is devoted to the proof of Theorem III.15: it relies on three lemmas. The first two lemmas are inspired by the work of Chouk and Friz [CF16], while the third lemma is inspired by the work of Hairer and Schönbauer [HS21].

We first introduce, for any function $h \in L^\infty_{p_{\kappa/2}}(\mathbb{R}^2)$, the shift operator T_h on $\mathscr{M}$ as follows:

$$T_h q = T_h(X, U) = (X + h, U + 2(\nabla G * X) \cdot (\nabla G * h) + |\nabla G * h|^2) \;. \tag{III.14}$$

One can check that $(h, q) \mapsto T_h q$ is continuous from $L^\infty_{p_{\kappa/2}}(\mathbb{R}^2) \times \mathscr{M}$ into $\mathscr{M}$ and that $T_h^{-1} = T_{-h}$.

Lemma III.17. *For all $\xi \in \Omega_0$ and all $h \in L^\infty_{p_{\kappa/2}}$, it holds $\xi + h \in \Omega_0$ and*

$$T_h Q(\xi) = Q(\xi + h) \;.$$

Proof. For any $\varepsilon \in (0,1)$, any $\xi \in \Omega$ and all $h \in L^\infty_{p_{a/2}}$ it holds

$$T_{h_\varepsilon} Q_\varepsilon(\xi) = Q_\varepsilon(\xi + h)$$

where $h_\varepsilon = h * \varrho_\varepsilon$ converges to h in $L^\infty_{p_{\kappa/2}}$ as $\varepsilon \to 0$. We now restrict ourselves to $\xi \in \Omega_0$. We know that $Q_{\varepsilon_k}(\xi)$ converges to $Q(\xi)$. The continuity of the shift operator thus implies that $T_{h_{\varepsilon_k}} Q_{\varepsilon_k}(\xi)$ converges to $T_h Q(\xi)$. We thus deduce that $Q_{\varepsilon_k}(\xi + h)$ converges. To conclude the proof, it suffices to show that $\xi + h \in \Omega_0$ so that the limit of $Q_{\varepsilon_k}(\xi + h)$ is necessarily equal to $Q(\xi + h)$ and therefore $T_h Q(\xi) = Q(\xi + h)$.
To check that $\xi + h \in \Omega_0$, by the definition of Ω_0 it suffices to show that for all $x \in \mathbb{R}^2$, $Q_{\varepsilon_k}(\theta_x(\xi + h))$ converges. The continuity of θ_x in $\mathscr{M}$ combined with the convergence of $Q_{\varepsilon_k}(\xi + h)$ and the identity $Q_{\varepsilon_k}(\theta_x(\xi + h)) = \theta_x Q_{\varepsilon_k}(\xi + h)$ allows to conclude. □

Recall that $b \in (0, \kappa/2)$.

Lemma III.18. *For any $q \in \mathtt{supp}(Q)$ and any $h \in L^\infty_{p_b}$, $T_h q \in \mathtt{supp}(Q)$.*

Proof. Suppose that the property holds true under the restrictive assumption that $h \in L^2 \cap L^\infty_{p_b}$. Fix $h \in L^\infty_{p_b}$. There exists $h_n \in L^2 \cap L^\infty_{p_b}$ such that $h_n \to h$ in $L^\infty_{p_{\kappa/2}}$ (for instance, take h_n to be the restriction of h to $[-n,n]^2$). The continuity of the shift operator then implies that $T_{h_n} q$ converges to $T_h q$. Since $\mathtt{supp}(Q)$ is a closed set, we conclude that $T_h q \in \mathtt{supp}(Q)$.
Let us now prove that the property holds for $h \in L^2 \cap L^\infty_{p_b}$. Assume that $q \in \mathtt{supp}(Q)$. For any open set $U \subset \mathscr{M}$ containing q, it holds $\mathbb{P}(Q \in U) > 0$. Fix some open set V containing $T_h q$. By continuity, the set $U := T_h^{-1} V = T_{-h} V$ is open and contains q. Therefore $\mathbb{P}(Q \in V) = \mathbb{P}(Q \in T_h U) = \mathbb{P}(T_{-h} Q \in U)$. By the previous lemma, $T_{-h} Q(\xi) = Q(\xi - h)$ for all $\xi \in \Omega_0$. The Cameron-Martin Theorem ensures that the law of $\xi \mapsto Q(\xi - h)$ is equivalent to the law of $Q(\xi)$. Since $\mathbb{P}(Q \in U) > 0$, we deduce that $\mathbb{P}(T_{-h} Q \in U) > 0$ and consequently $\mathbb{P}(Q \in V) > 0$. We have therefore proven that any open set V containing $T_h q$ has positive measure under the law of Q: necessarily $T_h q \in \mathtt{supp}(Q)$. □

Lemma III.19. *Fix $c \in \mathbb{R}$. There exist two functions $\delta \mapsto \lambda_\delta \in (0,1)$ and $\delta \mapsto a_\delta \in \mathbb{R}$ such that if for any $\xi \in \Omega$ we set*

$$h_\delta(\xi) := -\xi_\delta + a_\delta \xi_{\lambda_\delta} \ ,$$

then the $\mathscr{M}$-valued random variable $T_{h_\delta} Q(\xi)$ converges in probability to $(0, c)$.

Proof. Observe that, given any two functions $\delta \mapsto \lambda_\delta \in (0,1)$ and $\delta \mapsto a_\delta \in \mathbb{R}$, $h_\delta(\xi)$ belongs to $L^\infty_{p_b}$ so that $T_{h_\delta} Q(\xi)$ is well-defined. Note that $T_{h_\delta} Q(\xi) = (\xi + \tau_\delta, Y_\delta)$ for some r.v. Y_δ.
Suppose for now that $a_\delta \to 0$ as $\delta \to 0$. Then it is straightforward that $\xi + \tau_\delta$ converges to 0 in $\mathcal{C}^{-\kappa}_{p_\kappa}$ as required. Regarding the second argument, it writes

$$\begin{aligned} Y_\delta =& Z + |\nabla G * \xi_\delta|^2 - 2a_\delta (\nabla G * \xi_\delta) \cdot (\nabla G * \xi_\lambda) + a_\delta^2 |\nabla G * \xi_\lambda|^2 \\ &- 2(\nabla G * \xi_\delta) \cdot (\nabla G * \xi) + 2a_\delta (\nabla G * \xi_\lambda) \cdot (\nabla G * \xi). \end{aligned}$$

We will show that there exist some functions a_δ and λ_δ such that

$$C_\delta := \sup_{x\in\mathbb{R}^2}\sup_{\mu\in(0,1]}\sup_{\eta\in\mathcal{B}^r}\frac{\mathbb{E}[\langle Y_\delta - c, \eta_x^\mu\rangle^2]}{\mu^{-\kappa}} \to 0\,, \quad \delta\downarrow 0\,.$$

With this estimate at hand and following a similar computation as in [HL15, Proof of Proposition 1.3] one can obtain for m large enough

$$\mathbb{E}\,\|Y_\delta - c\|^{2m}_{\mathcal{C}^{-\kappa}_{p_\kappa}} \lesssim \sum_{x\in\mathbb{Z}^2} p_\kappa(x)^{-2m}\sum_{n\ge 0} 2^{2n}2^{-2nm\kappa}2^{nm\kappa}C_\delta^m \lesssim C_\delta^m\,,$$

so the desired convergence of Y to c follows.
Let us now prove the above estimate. Since Y is a stationary field, it suffices to consider $x=0$ in the estimate above. The Wiener chaos decomposition of $\langle Y_\delta - c, \eta^\mu\rangle$ yields two terms: its expectation, which lies in the 0-th Wiener chaos, and the rest, which lies in the 2-nd Wiener chaos. The second moment of this rest equals

$$\iint\Big(\int_{\mathbb{R}^2} I_\delta(x,z_1,z_2)\eta^\mu(x)\,\mathrm{d}x\Big)^2\mathrm{d}z_1\,\mathrm{d}z_2\,,$$

where

$$\begin{aligned}I_\delta(x,z_1,z_2) &= \nabla G(z_1-x)\cdot\nabla G(z_2-x) + \nabla G_\delta(z_1-x)\cdot\nabla G_\delta(z_2-x)\\ &\quad - 2a_\delta\nabla G_\delta(z_1-x)\cdot\nabla G_\lambda(z_2-x) + a_\delta^2\nabla G_\lambda(z_1-x)\cdot\nabla G_\lambda(z_2-x)\\ &\quad - 2\nabla G_\delta(z_1-x)\cdot\nabla G(z_2-x) + 2a_\delta\nabla G_\lambda(z_1-x)\cdot\nabla G(z_2-x)\,.\end{aligned}$$

Here $G_\delta := G*\varrho_\delta$. Let us show that, as soon as $\lambda_\delta\le\delta$ and $a_\delta\downarrow 0$ as $\delta\downarrow 0$,

$$\sup_{\mu\in(0,1)}\mu^\kappa\iint\Big(\int_{\mathbb{R}^2} I_\delta(x,z_1,z_2)\eta^\mu(x)\,\mathrm{d}x\Big)^2\mathrm{d}z_1\,\mathrm{d}z_2\,,$$

goes to 0 as $\delta\downarrow 0$. The above integral equals

$$\iint H_\delta(x-x')\eta^\mu(x)\eta^\mu(x')\,\mathrm{d}x\,\mathrm{d}x'$$

where $H_\delta(x,x') = \iint_{z_1,z_2} I_\delta(x,z_1,z_2)I_\delta(x',z_1,x_2)\,\mathrm{d}z_1\,\mathrm{d}z_2$. By expanding the integrand, one can write $H_\delta(x,x') = H_\delta^{(0)}(x-x') + \sum_{k=1}^4 a_\delta^k H_\delta^{(k)}(x-x')$ where the first term is given by

$$\begin{aligned}H_\delta^{(0)} = \sum_{i,j=1}^2\Big(&\partial_i(G-G_\delta)*\partial_j(G-G_\delta)\cdot\partial_i G*\partial_j G\\ &+ 2\,\partial_i G_\delta*\partial_j(G-G_\delta)\cdot\partial_i(G_\delta-G)*\partial_j G\\ &+ \partial_i G_\delta*\partial_j G_\delta\cdot\partial_i(G_\delta-G)*\partial_j(G_\delta-G)\Big)\end{aligned}$$

Given that the integral kernel G has a sigularity at 0 of order $-\kappa'$, for all $\kappa'>0$, [Hai14, Sec. 10] entails that $\iint H_\delta^{(0)}(x',x'')\eta^\mu(x)\eta^\mu(x')\,\mathrm{d}x\,\mathrm{d}x' \lesssim \delta^{\kappa'}\mu^{-5\kappa'}$. Similarly, for $k\ge 1$,

$H_\delta^{(k)}$ is an integral kernel of order $-4\kappa'$ and one has $\iint H_\delta^{(k)}(x-x')\eta^\mu(x)\eta^\mu(x') \lesssim \mu^{-4\kappa'}$. By choosing $5\kappa' < \kappa$ and since $a_\delta \to 0$, we then deduce $\mu^\kappa \iint H_\delta(x-x')\eta^\mu(x)\eta^\mu(x')\,\mathrm{d}x\,\mathrm{d}x' \lesssim \delta^{\kappa'} + O(a_\delta) \to 0$ uniformly for $\mu \in (0,1]$, proving the assertion.

To complete the proof, we will show that one can choose a_δ, λ_δ and a parameter δ_0 in such a way that the expectation of $\langle Y_\delta - c, \eta^\mu\rangle$ vanishes for all $\delta \in (0,\delta_0)$ and all η, μ. For all constants $\delta_1, \delta_2 \in (0,1)$, we introduce the notation

$$v(\delta_1,\delta_2) := \mathbb{E}\left[(\nabla G * \xi_{\delta_1})(0)\cdot(\nabla G * \xi_{\delta_2})(0)\right] = \int_{\mathbb{R}^2} \nabla G_{\delta_1}(x)\cdot\nabla G_{\delta_2}(x)\,\mathrm{d}x\ ,$$

and when either δ_1 or δ_2 equals 0, we replace ∇G_{δ_j} simply by ∇G in the expression. It can be checked that there exist $0 < c_1 < c_2$ such that for all δ_1, δ_2 as above

$$c_1 \log[(\delta_1 \vee \delta_2)^{-1}] \le v(\delta_1,\delta_2) \le c_2 \log[(\delta_1 \vee \delta_2)^{-1}]\ .$$

The expectation of $\langle Y_\delta - c, \eta^\mu\rangle$ then equals

$$\int_{\mathbb{R}^2} \Big(v(\delta,\delta) - 2a_\delta v(\delta,\lambda) + a_\delta^2 v(\lambda,\lambda) - 2v(\delta,0) + 2a_\delta v(\lambda,0) - c\Big)\eta^\mu(x)\,\mathrm{d}x\ .$$

We now set $\lambda = \delta^{\delta^{-1}}$. The proof is complete if we can show there exists a parameter $\delta_0 \in (0,1)$ and a function a_δ satisfying $|a_\delta| \le \delta$ and

$$c = v(\delta,\delta) - 2a_\delta v(\delta,\lambda) + a_\delta^2 v(\lambda,\lambda) - 2v(\delta,0) + 2a_\delta v(\lambda,0)\ ,\quad \forall \delta \in (0,\delta_0)\ .$$

We proceed through a fixed point argument. For $\delta_0 \in (0,1)$, define F_{δ_0} as the metric space of all continuous functions $f : (0,\delta_0) \to \mathbb{R}$ such that

$$\|f\|_{\delta_0} = \sup_{\delta \in (0,\delta_0)} \frac{|f(\delta)|}{\delta} < \infty\ .$$

For $R > 0$, let $F_{\delta_0,R}$ be the subset of all f such that $\|f\|_{\delta_0} \le R$. For any $f \in F_{\delta_0}$ define

$$Mf(\delta) := \frac{1}{2h(\lambda,0)}\Big(c - h(\delta,\delta) + 2f(\delta)h(\delta,\lambda) - f(\delta)^2 h(\lambda,\lambda) + 2h(\delta,0)\Big)$$

Since $\log(\lambda^{-1}) = \delta^{-1}\log(\delta^{-1})$, one can check that, for all $\delta_0 \in (0,1)$, M maps F_{δ_0} into itself. Furthermore, there exists $R > 0$ such that for all δ_0 small enough, M maps $F_{\delta_0,R}$ into itself.
Moreover, there exists a constant $C > 0$ such that for any two elements $f, g \in F_{\delta_0,R}$, we have

$$\begin{aligned}\|Mf - Mg\|_{\delta_0} &\le \|f-g\|_{\delta_0} \sup_{\delta\in(0,\delta_0)} \left|\frac{2|h(\delta,\lambda)| + |f(\delta)+g(\delta)||h(\lambda,\lambda)|}{2h(\lambda,0)}\right| \\ &\le C\delta_0(1+R)\,\|f-g\|_{\delta_0}\ .\end{aligned}$$

Choose δ_0 such that $C\delta_0(1+R) < 1$. Then M admits a unique fixed point $a \in F_{\delta_0,R}$. This suffices to conclude the proof. □

Proof of Theorem III.15. Set $A := \overline{\{(h, |\nabla G * h|^2 + c) : h \in L^\infty_{p_b}(\mathbb{R}^2)\ ,\ c \in \mathbb{R}\}}^{\mathscr{M}}$. For all $\varepsilon \in (0,1)$, the r.v. Q_ε takes values in the closed set A: consequently $\mathtt{supp}(Q_\varepsilon) \subset A$. Since Q is the limit in probability of Q_ε, we deduce that $\mathtt{supp}(Q) \subset A$. It remains to prove the converse inclusion.
Fix $c \in \mathbb{R}$. Assume that $(0,c) \in \mathtt{supp}(Q)$. Then Lemma III.18 implies that for all $h \in L^\infty_{p_b}$, $T_h(0,c) \in \mathtt{supp}(Q)$ and the desired inclusion follows. It remains to show that $(0,c) \in \mathtt{supp}(Q)$.
Using the notations of Lemma III.19, for any $\delta \in (0,1)$, we set

$$h_\delta(\xi) := -\xi_\delta + a_\delta \xi_{\lambda_\delta}\ ,$$

By this lemma, there exists $\delta_n \downarrow 0$ such that on some event Ω_0' of full $\mathbb{P}$-measure, $T_{h_{\delta_n}(\xi)}Q(\xi)$ converges to $(0,c)$ for all $\xi \in \Omega_0'$.
Fix $q \in \mathtt{supp}(Q) \cap Q(\Omega_0')$. There exists $\xi \in \Omega_0'$ such that $q = Q(\xi)$. By Lemma III.18, $T_{h_{\delta_n}(\xi)}Q(\xi) \in \mathtt{supp}(Q)$. Since the support is a closed set, we deduce that $(0,c) \in \mathtt{supp}(Q)$. □

III.4 THE ANDERSON HAMILTONIAN IN DIMENSION THREE

We follow the same strategy as in dimension two. However, the construction of the (PAM) in dimension three

$$\begin{cases} \partial_t u = \Delta u - \xi u, & t > 0,\ x \in \mathbb{R}^3, \\ u(t=0,\cdot) = f \end{cases} \tag{III.15}$$

is much harder as the noise is more singular, we thus rely on [HL18] which performed its construction with the theory of regularity structures [Hai14].
In this whole section, $\kappa > 0$ is a small parameter (it is implicitly taken as small as needed). Furthermore we set

$$\gamma := \frac{3}{2} + 2\kappa\ , \quad \alpha := -\frac{3}{2} - \kappa\ .$$

III.4.1 THE SPACE OF ENHANCED NOISES AND THE WHITE NOISE CASE

A regularity structure is a triplet $(\mathcal{A}, \mathcal{T}, \mathcal{G})$ satisfying the following properties:

1. $\mathcal{A} \subset \mathbb{R}$ is a locally finite set of indices that is bounded from below and contains 0.
2. $\mathcal{T} = \bigoplus_{\alpha \in \mathcal{A}} \mathcal{T}_\alpha$ is a graded vector space, where for each $\alpha \in \mathcal{A}$, $\mathcal{T}_\alpha$ is a finite-dimensional Banach space equipped with the norm $\|\cdot\|_\alpha$. We impose $\mathcal{T}_0 \simeq \mathbb{R}$ with the unit vector $\mathbf{1}$. For any vector τ in a finite-dimensional subspace of $\mathcal{T}$, we write $\|\tau\|$ for its Euclidean norm.
3. $\mathcal{G}$ is a group acting on $\mathcal{T}$ such that every element Γ of $\mathcal{G}$ satisfies $\Gamma\mathbf{1} = \mathbf{1}$ and for all $\tau \in \mathcal{T}_\alpha$, $\Gamma\tau - \tau \in \mathcal{T}_{<\alpha} := \bigoplus_{\beta \in \mathcal{A}_{<\alpha}} \mathcal{T}_\beta$ where $\mathcal{A}_{<\alpha} = \mathcal{A} \cap (-\infty, \alpha)$.

$\mathcal{U}$	$\mathcal{A}(\mathcal{U})$	$\mathcal{F}$	$\mathcal{A}(\mathcal{F})$
$\mathbf{1}$	0	Ξ	$-\frac{3}{2}-\kappa$
$\mathcal{I}(\Xi)$	$\frac{1}{2}-\kappa$	$\Xi\mathcal{I}(\Xi)$	$-1-2\kappa$
$\mathcal{I}(\Xi\mathcal{I}(\Xi))$	$1-2\kappa$	$\Xi\mathcal{I}(\Xi\mathcal{I}(\Xi))$	$-\frac{1}{2}-3\kappa$
X_i	1	ΞX_i	$-\frac{1}{2}-\kappa$
$\mathcal{I}(\Xi\mathcal{I}(\Xi\mathcal{I}(\Xi)))$	$\frac{3}{2}-3\kappa$	$\Xi\mathcal{I}(\Xi\mathcal{I}(\Xi\mathcal{I}(\Xi)))$	-4κ
$\mathcal{I}(\Xi X_i)$	$\frac{3}{2}-\kappa$	$\Xi\mathcal{I}(\Xi X_i)$	-2κ

Figure III.1: Basis vectors and homogeneities in the regularity structure.

A basic example is the polynomial regularity structure where $\mathcal{A}=\mathbb{N}$, and for each $n\in\mathbb{N}$, $\mathcal{T}_n$ is the vector space spanned by all $X^k := X_1^{k_1}X_2^{k_2}X_3^{k_3}$, $k\in\mathbb{N}^3$ such that $|k| := k_1+k_2+k_3=n$ and $\mathcal{G}$ is the group formed by all the transformations Γ_h that translate any polynomial P by the vector h: $\Gamma_h P(X)=P(X+h)$.

Let us now introduce the regularity structure associated to the (PAM). Let Ξ be a formal expression (aimed at representing ξ at the level of the model space $\mathcal{T}$). Let $\mathcal{F}$ and $\mathcal{U}$ be the smallest sets of formal expressions such that $\mathcal{F}$ contains Ξ, $\mathcal{U}$ contains all polynomials X^k and

$$\tau\in\mathcal{U} \implies \tau\Xi\in\mathcal{F} \quad\text{and}\quad \tau\in\mathcal{F}\implies \mathcal{I}(\tau)\in\mathcal{U}.$$

Here $\tau\Xi$ and $\mathcal{I}(\tau)$ denote new formal expressions that need to be considered. The formal expressions in $\mathcal{U}$ will be used in the local description of the solution u while those in $\mathcal{F}$ will be used for the product $u\xi$.

Each expression τ is assigned a number $|\tau|$ called homogeneity, which is calculated by the following rules: (1) $|X^k|=|k|$, (2) $|\Xi|=\alpha$, (3) for any τ,τ', $|\tau\tau'|=|\tau|+|\tau'|$, (4) for any τ, $|\mathcal{I}(\tau)|=|\tau|+2$.
We thus set $\mathcal{A} := \{|\tau| : \tau\in\mathcal{F}\cup\mathcal{U}\}$ and for $\alpha\in\mathcal{A}$, we let $\mathcal{T}_\alpha$ be the vector space spanned by all $\tau\in\mathcal{F}\cup\mathcal{U}$ such that $|\tau|=\alpha$. For the construction of the structure group $\mathcal{G}$, we refer the reader to [Hai14, Sec. 8.1].

From now on, we will *always* restrict $\mathcal{U}$ (resp. $\mathcal{F}$) to formal expressions whose homogeneities are below γ (resp. $\gamma+\alpha$), see Figure III.1.

We also consider the notion of admissible models as introduced in [HL18, Definition 2.2] associated to a compactly supported kernel $\bar{K}_+$ which coincides near 0 with the Green's function of the Laplacian in dimension 3. We will denote by $q=(\Pi,\Gamma)$ a generic admissible model.

Remark III.20. In [HL18], admissible models are in space-time and are taken w.r.t. the kernel P_+, which is a space-time function that coincides with the heat kernel near the origin: the setting therein was designed to encompass simultaneously the multiplicative (SHE) in dimension 1 (for which the noise is space-time) and the (PAM) for which the noise is in space only. In the latter case, that we consider in this article, the stochastic objects only evolve in space so that the model is in space only and the kernel $\bar{K}_+$ is nothing but the integral in time of P_+.

We recall the notation

$$\|\Pi\|_x := \sup_{\varphi\in\mathcal{B}_r}\sup_{\lambda\in(0,1]}\sup_{\substack{\zeta\in\mathcal{A}\\ \tau\in\mathcal{T}_\zeta}} \frac{|\langle \Pi_x\tau,\varphi_x^\lambda\rangle|}{\lambda^\zeta\|\tau\|}\ ,\ \|\Gamma\|_{x,y} := \sup_{\substack{\beta\le\zeta\in\mathcal{A}\\ \tau\in\mathcal{T}_\zeta}} \frac{|\Gamma_{x,y}\tau|_\beta}{|x-y|^{\zeta-\beta}\|\tau\|}\ .$$

The set of all admissible models $q=(\Pi,\Gamma)$ is endowed with the distance

$$\|q;\bar q\| := \sup_{x\in\mathbb{R}^3}\frac{\|\Pi-\bar\Pi\|_x}{p_\kappa(x)} + \sup_{x,y\in\mathbb{R}^3:|x-y|\le 1}\frac{\|\Gamma-\bar\Gamma\|_{x,y}}{p_\kappa(x)}\ .$$

It turns out that for admissible models the Γ can be read off the Π: more precisely

$$|||q;\bar q||| := \sup_{x\in\mathbb{R}^3}\sup_{\varphi\in\mathcal{B}_r}\sup_{\lambda\in(0,1]}\sup_{\tau\in\mathcal{A}} \frac{|\langle \Pi_x\tau-\bar\Pi_x\tau,\varphi_x^\lambda\rangle|}{\lambda^{|\tau|}p_\kappa(x)}\ , \tag{III.16}$$

defines an equivalent metric on the set of admissible models, see for instance [HW15, Sec 2.4].
An admissible model $q=(\Pi,\Gamma)$ is called *smooth* if $\Pi_x\tau\in C^\infty(\mathbb{R}^3)$ for all $x\in\mathbb{R}^3$ and all $\tau\in\mathcal{A}$. We then let $\mathscr{M}$ be the closure of all smooth models in the set of admissible models, it is a complete and separable metric space.

We fix some parameter $b\in(0,\kappa/4)$. Given some function $h\in C^\infty_{p_b}(\mathbb{R}^3)$, one can define the canonical model $q_{\mathrm{can}}(h)$ associated to h: this is the only admissible model satisfying

$$\Pi^{\mathrm{can}}(h)_x(\Xi)(y) = h(y) \text{ and } \Pi^{\mathrm{can}}(h)_x(\tau\bar\tau)(y) = \Pi^{\mathrm{can}}(h)_x(\tau)(y)\Pi^{\mathrm{can}}(h)_x(\bar\tau)(y)\ .$$

Let us mention that we took b smaller than $\kappa/4$ in order for the norm of this model to be finite: indeed 4 is the largest number of occurrences of Ξ in the basis vectors of the regularity structure at stake and $p_b^4\le p_\kappa$.
In [HP15, Section 4.3], a three dimensional renormalisation group was identified for the regularity structure of the generalised (PAM): it applies in particular to the regularity structure of (PAM). The definition is very lengthy so let us simply recall that, given any triplet $\bar c=(c,c^{(1,1)},c^{(1,2)})\in\mathbb{R}^3$, there exists a *renormalised* canonical model $\mathscr{R}^{\bar c}q_{\mathrm{can}}(h) = (\hat\Pi(h),\hat\Gamma(h))\in\mathscr{M}$ and it satisfies

$$\hat\Pi(h)_x\Xi\mathcal{I}(\Xi)(x) = -c\ ,\quad \hat\Pi(h)_x\Xi\mathcal{I}(\Xi\mathcal{I}(\Xi\mathcal{I}(\Xi)))(x) = -c^{(1,1)}-c^{(1,2)}\ ,\quad x\in\mathbb{R}^3\ .$$

We set $\Omega := \mathcal{C}^{-\frac{3}{2}-\kappa}_{p_b}(\mathbb{R}^3)$ endowed with the law $\mathbb{P}$ of white noise. The canonical variable on Ω will be denoted ξ. Following [HL18, Theorem 5.3], we consider the renormalised canonical model $Q_\varepsilon(\xi) = \mathscr{R}^{\bar c_\varepsilon}q_{\mathrm{can}}(\xi_\varepsilon)$ associated to the smooth noise $\xi_\varepsilon = \xi*\varrho_\varepsilon$ and the renormalisation constants $c_\varepsilon = c_\varrho\varepsilon^{-1}+O(1)$, $c_\varepsilon^{(1,1)} = \ln\varepsilon^{-1}+O(1)$ and $c_\varepsilon^{(1,2)} = O(1)$.
The r.v. Q_ε converges in $\mathscr{M}$ to some limit Q in probability, see [HL18, Theorem 5.3]. Actually, in finite volume, this result was proven by Hairer and Pardoux [HP15], its

extension to infinite volume rests on the same arguments as in (III.7) with ξ_ε replaced by $Q - Q_\varepsilon$: all the stochastic objects live in finite Wiener chaoses and therefore admit finite moments.

To identify the spectrum of $\mathcal{H}$ in section III.5, we will need to show a commutation property: the operator $\mathcal{H}(\theta_x\xi)$ associated to the shifted noise coincides with the operator $\mathcal{H}(\xi)$ conjugated with spatial shifts, see Lemma III.30. One therefore needs to construct the shifted model for *all* $x \in \mathbb{R}^3$ simultaneously. This is the purpose of the next result.

Lemma III.21. *There exists a sequence $\varepsilon_k \downarrow 0$ such that the set*

$$\Omega_0 := \Big\{\xi \in \Omega : Q_{\varepsilon_k}(\theta_x\xi) \text{ converges in } \mathscr{M} \text{ as } k \to \infty\ , \quad \forall x \in \mathbb{R}^3\Big\}\ , \tag{III.17}$$

is of full $\mathbb{P}$-measure and is invariant under all $\theta_x, x \in \mathbb{R}^3$. The[2] limit Q satisfies $Q(\theta_x\xi) = \theta_x Q(\xi)$ for all $\xi \in \Omega_0$.
As $\varepsilon \downarrow 0$, the field $(Q_\varepsilon(\theta_x\xi))_{x\in\mathbb{R}^3}$ which takes values in $\mathscr{M}$ converges in probability, locally uniformly over $x \in \mathbb{R}^3$, to $(Q(\theta_x\xi))_{x\in\mathbb{R}^3}$.

Proof. It is straightforward to check that for any $\varepsilon \in (0,1)$, any $x, y \in \mathbb{R}^3$ and any $\xi \in \Omega$

$$\langle \Pi_y^{(\varepsilon)}(\theta_x\xi)\tau, \varphi_y\rangle = \langle \Pi_{y-x}^{(\varepsilon)}(\xi)\tau, \varphi_{y-x}\rangle\ .$$

This observation combined with the definition of the metric (III.16) leads to

$$\interleave Q_\varepsilon(\theta_x\xi); Q_{\varepsilon'}(\theta_x\xi)\interleave \le \sup_{y\in\mathbb{R}^3} \frac{p_\kappa(y-x)}{p_\kappa(y)} \interleave Q_\varepsilon(\xi); Q_{\varepsilon'}(\xi)\interleave = p_\kappa(x)\interleave Q_\varepsilon(\xi); Q_{\varepsilon'}(\xi)\interleave\ ,$$

for all $\xi \in \Omega$ and all $x \in \mathbb{R}^d$. The arguments in [HL18, Theorem 5.3] show that, for any given $p > 1$, $Q_\varepsilon(\xi)$ converges in $L^p(\Omega,\mathbb{P})$ as $\varepsilon \downarrow 0$ to some limit $Q(\xi)$ in $\mathscr{M}$. We thus deduce that the collection of r.v. $(Q_\varepsilon(\theta_x\xi), x \in \mathbb{R}^3)$, taking values in $L^\infty_{p_\kappa}(\mathbb{R}^3, \mathscr{M})$, converges as $\varepsilon \downarrow 0$ in $L^p(\Omega,\mathbb{P})$. Therefore there exists a deterministic sequence $\varepsilon_k \downarrow 0$ such that the convergence holds almost surely. We thus consider the set of full $\mathbb{P}$-measure

$$\Omega_0 := \Big\{\xi \in \Omega : Q_{\varepsilon_k}(\theta_x\xi) \text{ converges in } \mathscr{M} \text{ as } k \to \infty\ , \quad \forall x \in \mathbb{R}^2\Big\}\ .$$

We can conclude the proof using the exact same arguments as in dimension 2, see the proof of Lemma III.4. □

III.4.2 The parabolic evolution

Theorem III.22. *There exists a map $(q, f, t) \mapsto u^{q,f}(t,\cdot)$ defined on $\mathscr{M} \times \Big(\bigcup_{\ell_0\in\mathbb{R}} L^2_{e_{\ell_0}}\Big) \times \mathbb{R}_+$ such that for any $\ell_0 \in \mathbb{R}$ the following properties hold:*

[2] Q is defined as a limit on the set of full measure Ω_0, and is extended to Ω arbitrarily.

1. *(Continuity in the data) For any $t > 0$, the map $(q, f) \mapsto u^{q,f}(t, \cdot)$ is continuous from $\mathscr{M} \times L^2_{e_{\ell_0}}$ into $L^2_{e_{\ell_0+t}}$, and is linear w.r.t. f.*
2. *(Continuity in time) For any $(q, f) \in \mathscr{M} \times L^2_{e_{\ell_0}}$ and any $T > 0$, the map $t \mapsto u^{q,f}(t, \cdot)$ is continuous from $[0, T]$ to $L^2_{e_{\ell_0+T}}$.*
3. *(Semigroup property) For any $(q, f) \in \mathscr{M} \times L^2_{e_{\ell_0}}$ and any $0 \le s \le t$, $u^{q,f}(t, \cdot)$ coincides with $u^{q,g}(s, \cdot)$ where $g = u^{q,f}(t - s, \cdot)$.*
4. *(Symmetry) For any $q \in \mathscr{M}$, $f, g \in C_c$ and any $t \ge 0$, it holds*
$$\langle f, u^{q,g}(t)\rangle = \langle u^{q,f}(t), g\rangle \ .$$
5. *(Classical solution) When $q = Q_\varepsilon(\xi)$ for some $\xi \in \Omega$ and $f \in L^2_{e_{\ell_0}}$, $u^{q,f}$ coincides with the classical solution to* (III.15) *with noise term $\xi_\varepsilon + C_\varepsilon$.*

The rest of this subsection is devoted to the proof of this theorem. We use the material from [HL18]. Recall that $\kappa > 0$ is small parameter, that $\gamma = \frac{3}{2} + 2\kappa$, $\alpha = -\frac{3}{2} - \kappa$ and set $\eta = -\frac{1}{2} + 3\kappa$. Let us mention that η is the worst regularity allowed for the initial condition (we do not need such a bad regularity in the present context). It is proven in [HL18, Theorem 5.2] that, for any given $T > 0$, there is a unique solution u in some space $\mathscr{D}^{\gamma,\eta,2}_{w,T}$ (whose definition can be found in [HL18, Definition 3.7] but this is unimportant for the moment) to the fixed point problem

$$\mathrm{u} = (\mathcal{P}_+ + \mathcal{P}_-)(\mathrm{u}\Xi) + \mathcal{P} f \ ,$$

and that this solution is continuous w.r.t. the pair (q, f). By [HL18, Theorem 4.3 and Lemma 4.6], it holds

$$\mathcal{R}\Big((\mathcal{P}_+ + \mathcal{P}_-)(\mathrm{u}\Xi)\Big) = K * \mathcal{R}(\mathrm{u}\Xi) \ ,$$

where K is the three-dimensional heat kernel. The arguments below show that $\mathcal{R}(\mathrm{u}\Xi)$ can be seen as a continuous function of time valued in a space of distrubutions on $\mathbb{R}^3$: this will allow us to write the convolution in space-time as an integral in time of a convolution in space, see (III.19), and then to apply the very same arguments as in dimension 2 to deduce the properties listed in the statement of the theorem.

For $\beta < 0$, $\ell \in \mathbb{R}$ and $\delta > 0$, let $\mathcal{B}^\beta_{\ell,\delta}$ be the Banach space of all distributions g on $\mathbb{R}^3$ such that

$$\|g\|_{\mathcal{B}^\beta_{\ell,\delta}} := \sup_{\lambda\in(0,1]} \Bigg\| \sup_{\varphi\in\mathcal{B}^r(\mathbb{R}^3)} \frac{|\langle g, \varphi^\lambda_x\rangle|}{e_\ell(x) p_\delta(x) \lambda^\beta} \Bigg\|_{L^2(\mathbb{R}^3,\mathrm{d}x)} < \infty \ . \tag{III.18}$$

This space is nothing but a weighted Besov space with parameters $p = 2$ and $q = \infty$. Given $\ell_0, \nu \in \mathbb{R}$ and $T > 0$, let $\mathscr{E}^{\beta,\nu}_{\ell_0,\delta,T}$ be the Banach space of all functions g on $(0, T)$ valued in the space of distributions on $\mathbb{R}^3$ such that

$$\|g\|_{\mathscr{E}^{\beta,\nu}_{\ell_0,\delta,T}} := \sup_{t\in(0,T)} \frac{\|g(t)\|_{\mathcal{B}^\beta_{\ell_0+t,\delta}}}{t^{\frac{\nu}{2}}} + \sup_{t\in(0,T):s\in(t/2,t)} \frac{\|g(t) - g(s)\|_{\mathcal{B}^\beta_{\ell_0+t,\delta}}}{t^{\frac{\nu}{2}} |t - s|^{\kappa/4}} < \infty \ .$$

We now claim that there exists some constant $\delta > 0$ such that for all $T > 0$

$$(q, f) \mapsto \mathcal{R}(\mathrm{u}\Xi) \;,$$

is a continuous map from $\mathscr{M} \times L^2_{e_{\ell_0}}$ into $\mathscr{E}^{\alpha,\eta+\alpha-\kappa/2}_{\ell_0,\delta,T}$. Assuming that this claim holds, we deduce that

$$u^{q,f}(t) := \mathcal{R}(\mathrm{u})(t) = \int_0^t K(t-s) * \mathcal{R}(\mathrm{u}\Xi)(s)ds + K(t) * f \;, \quad t > 0 \;. \tag{III.19}$$

Then the very same arguments as in dimension 2 allow to deduce Properties 1., 2. and 3. On the other hand, Property 5. was already proven in [HL18, Theorem 5.3]. Regarding Property 4., the continuity properties already proven show that one can restrict oneself to a smooth model q. In this context, there exists a smooth function ζ that only depends on the model q which is such that

$$\mathcal{R}(\mathrm{u}\Xi)(t,x) = u(t,x)\zeta(x) \;.$$

If we let $u^f(t,x), u^g(t,x)$ be the solutions starting from f, g then these two functions are strong solutions to the PDE

$$\partial_t u = \Delta u + u\zeta \;,$$

starting from f and g respectively. Now if we set

$$h(s) := \langle u^f(t-s), u^g(s)\rangle \;, \quad s \in [0,t] \;,$$

then the desired property can be written as $h(0) = h(t)$. However h is a continuous function and an integration by parts shows that $h'(s) = 0$ for all $s \in (0,t)$.

We are left with the proof of the claim. A key argument is the following version of the reconstruction theorem. Fix $\tilde\gamma > 0$, $\tilde\eta < \alpha$, $\delta > 0$ and $t > 0$. Given a model q, we let $\mathscr{D}^{\tilde\gamma,\tilde\eta}_{\ell_0,\delta,t}(q)$ be the set of maps h from $\mathbb{R}^3$ into the regularity structure satisfying

$$\begin{aligned}\|\mathrm{h}\|_{\tilde\gamma,\tilde\eta,\ell_0,\delta,t} := \sum_{\zeta\in\mathcal{A}_{<\tilde\gamma}} \Bigg(\Bigg\| \frac{|\mathrm{h}(x)|_\zeta}{t^{\frac{\tilde\eta-\zeta}{2}} e_{\ell_0+t}(x) p_\delta(x)} \Bigg\|_{L^2} \\ + \sup_{\lambda\in(0,\sqrt{t/2}]} \Bigg\| \int_{y\in B(x,\lambda)} \lambda^{-3} \frac{|\mathrm{h}(y) - \Gamma_{y,x}\mathrm{h}(x)|_\zeta}{t^{\frac{\tilde\eta-\tilde\gamma}{2}} e_{\ell_0+t}(x)p_\delta(x)\lambda^{\tilde\gamma-\zeta}} dy \Bigg\|_{L^2} \Bigg) \;.\end{aligned}$$

Given two models $q, \bar q$, we introduce the distance

$$\begin{aligned}\|\mathrm{h};\bar{\mathrm{h}}\|_{\tilde\gamma,\tilde\eta,\ell_0,\delta,t} := \sum_{\zeta\in\mathcal{A}_{<\tilde\gamma}} \Bigg(\Bigg\| \frac{|\mathrm{h}(x) - \bar{\mathrm{h}}(x)|_\zeta}{t^{\frac{\tilde\eta-\zeta}{2}} e_{\ell_0+t}(x) p_\delta(x)} \Bigg\|_{L^2} \\ + \sup_{\lambda\in(0,\sqrt{t/2}]} \Bigg\| \int_{y\in B(x,\lambda)} \lambda^{-3} \frac{|\mathrm{h}(y) - \bar{\mathrm{h}}(y) - \Gamma_{y,x}\mathrm{h}(x) + \bar\Gamma_{y,x}\bar{\mathrm{h}}(x)|_\zeta}{t^{\frac{\tilde\eta-\tilde\gamma}{2}} e_{\ell_0+t}(x)p_\delta(x)\lambda^{\tilde\gamma-\zeta}} dy \Bigg\|_{L^2} \Bigg) \;.\end{aligned}$$

Proposition III.23. *For any $t > 0$ and any model q, there exists a unique continuous linear map $\mathcal{R}_t = \mathcal{R}_t(q)$ from $\mathscr{D}^{\tilde\gamma,\tilde\eta}_{\ell_0,\delta,t}(q)$ into $\mathcal{B}^\alpha_{\ell_0+t,\delta+2a}$ that satisfies*

$$\|\mathcal{R}_t \mathrm{h}\|_{\mathcal{B}^\alpha_{\ell_0+t,\delta+2a}} \lesssim t^{\frac{\tilde\eta}{2}}(\|q\| + \|q\|^2)\|\mathrm{h}\|_{\tilde\gamma,\tilde\eta,\ell_0,\delta,t}\;, \tag{III.20}$$

as well as

$$\Big\| \sup_{\eta\in\mathcal{B}_r} \frac{\big|\langle \mathcal{R}_t\mathrm{h} - \Pi_x\mathrm{h}(x), \eta_x^\lambda\rangle\big|}{e_{\ell_0+t}(x)p_{\delta+2a}(x)}\Big\|_{L^2} \lesssim \lambda^{\tilde\gamma} t^{\frac{\tilde\eta-\tilde\gamma}{2}}(\|q\| + \|q\|^2)\|\mathrm{h}\|_{\tilde\gamma,\tilde\eta,\ell_0,\delta,t}\;, \tag{III.21}$$

uniformly over all $\lambda \in (0, \sqrt{t/2}]$, all $t > 0$ and all models q. Furthermore

$$\Big\| \sup_{\eta\in\mathcal{B}_r} \frac{\big|\langle \mathcal{R}_t\mathrm{h} - \bar{\mathcal{R}}_t\bar{\mathrm{h}} - \Pi_x\mathrm{h}(x) + \bar\Pi_x\bar{\mathrm{h}}(x), \eta_x^\lambda\rangle\big|}{e_{\ell_0+t}(x)p_{\delta+2a}(x)}\Big\|_{L^2}$$
$$\lesssim \lambda^{\tilde\gamma} t^{\frac{\tilde\eta-\tilde\gamma}{2}}\Big((\|q\| + \|q\|^2)\|\mathrm{h};\bar{\mathrm{h}}\|_{\tilde\gamma,\tilde\eta,\ell_0,\delta,t} + \|q-\bar q\|(1+\|q\|+\|\bar q\|)\|\bar{\mathrm{h}}\|_{\tilde\gamma,\tilde\eta,\ell_0,\delta,t}\Big)\;,$$

uniformly over all $\lambda \in (0, \sqrt{t/2}]$, and over all $t > 0$ and all models $q, \bar q$.

Proof. The reconstruction theorem stated in [HL18, Theorem 2.10] deals with space-time modelled distributions but its proof applies verbatim to the case of modelled distributions in space only. It yields the bound

$$\Big\| \sup_{\eta\in\mathcal{B}_r} \big|\langle \mathcal{R}_t\mathrm{h} - \Pi_x\mathrm{h}(x), \eta_x^\lambda\rangle\big|\Big\|_{L^2(B(x_0,1),dx)}$$
$$\lesssim \lambda^{\tilde\gamma} t^{\frac{\tilde\eta-\tilde\gamma}{2}}(p_a(x_0) + p_a(x_0)^2)e_{\ell_0+t}(x_0)p_\delta(x_0)(\|q\| + \|q\|^2)\|\mathrm{h}\|_{\tilde\gamma,\tilde\eta,\ell_0,\delta,t,x_0}\;,$$

uniformly over all $\lambda \in (0, \sqrt{t/2}]$, all $x_0 \in \mathbb{R}^3$. Here $\|\mathrm{h}\|_{\tilde\gamma,\tilde\eta,\ell_0,\delta,t,x_0}$ is obtained from the expression of $\|\mathrm{h}\|_{\tilde\gamma,\tilde\eta,\ell_0,\delta,t}$ by restricting the L^2 norm to the spatial domain $B(x_0, 2)$. We deduce the bound (III.21) of the statement. The case of two models follows as a straightforward adaptation.

Let us prove that $\mathcal{R}_t$ takes values in $\mathcal{B}^\alpha_{\ell_0+t,\delta+2a}$ and satisfies (III.20). We observe that uniformly over all $\lambda \le \sqrt{t/2}$

$$\Big\| \sup_{\eta\in\mathcal{B}_r} \frac{\big|\langle \Pi_x\mathrm{h}(x), \eta_x^\lambda\rangle\big|}{t^{\frac{\tilde\eta}{2}}\lambda^\alpha e_{\ell_0+t}(x)p_{\delta+2a}(x)}\Big\|_{L^2} \lesssim \|q\|\sum_\zeta \frac{\lambda^\zeta t^{\frac{\tilde\eta-\zeta}{2}}}{\lambda^\alpha t^{\frac{\tilde\eta}{2}}}\Big\|\frac{p_a(x)\big|\mathrm{h}(x)\big|_\zeta}{e_{\ell_0+t}(x)p_{\delta+2a}(x)}\Big\|_{L^2}\;,$$

Recall that $\zeta \ge \alpha$ hence the bound $\lambda^{\zeta-\alpha}t^{\frac{\tilde\eta-\zeta}{2}} \lesssim t^{\frac{\tilde\eta-\alpha}{2}} \lesssim t^{\frac{\tilde\eta}{2}}$, and we deduce that the last expression is bounded by a term of order $\|q\|\|\mathrm{h}\|_{\tilde\gamma,\tilde\eta,\ell_0,\delta,t}$ as required. Together with the reconstruction bound, this shows that

$$\Big\| \sup_{\eta\in\mathcal{B}_r} \frac{\big|\langle \mathcal{R}_t\mathrm{h}, \eta_x^\lambda\rangle\big|}{t^{\frac{\tilde\eta}{2}}\lambda^\alpha e_{\ell_0+t}(x)p_{\delta+2a}(x)}\Big\|_{L^2} \lesssim \|\mathrm{h}\|_{\tilde\gamma,\tilde\eta,\ell_0,\delta,t}\;,$$

uniformly over all $\lambda \in (0, \sqrt{t/2})$. On the other hand, for $\lambda \in [\sqrt{t/2}, 1]$, one can write

$$\eta_x^\lambda(\cdot) = \sum_{y\in(\lambda_0\mathbb{Z})^3} \eta_x^\lambda(\cdot)\psi((\cdot - y)/\lambda_0) \;,$$

where $\sum_{y\in\mathbb{Z}^3}\psi(\cdot - y)$ is a smooth, compactly supported partition of unity, and $\lambda_0 = \sqrt{t}$. The number of non-zero terms in this sum is of order $(\lambda/\lambda_0)^3$, and each $\eta_x^\lambda(\cdot)\psi((\cdot - y)/\lambda_0)$ can be seen as a function $(\lambda_0/\lambda)^3\varphi_y^{\lambda_0}$ where $\varphi \in \mathcal{B}_r$. By Jensen's inequality, we thus get

$$\begin{aligned}\Big|\langle \mathcal{R}_t \mathrm{h}, \eta_x^\lambda\rangle\Big|^2 &= \Big|\sum_{y\in(\lambda_0\mathbb{Z})^3} \langle \mathcal{R}_t \mathrm{h}, \eta_x^\lambda \psi((\cdot - y)/\lambda_0)\rangle\Big|^2\\ &\lesssim \sum_{y\in(\lambda_0\mathbb{Z})^3 : |y-x|\le\lambda} (\lambda_0/\lambda)^3 \sup_{\varphi\in\mathcal{B}_r} \Big|\langle \mathcal{R}_t \mathrm{h}, \varphi_y^{\lambda_0}\rangle\Big|^2 \;.\end{aligned}$$

Since φ_y can be seen as some function ψ_z with $z \in B(y,1)$, we obtain

$$\Big\| \sup_{\eta\in\mathcal{B}_r} \frac{\Big|\langle \mathcal{R}_t \mathrm{h}, \eta_x^\lambda\rangle\Big|}{t^{\frac{\tilde\eta}{2}}\lambda^\alpha e_{\ell_0+t}(x) p_{\delta+2a}(x)}\Big\|_{L^2} \lesssim \bigg(\int_{z\in\mathbb{R}^3} \sup_{\psi\in\mathcal{B}_r}\bigg(\frac{\Big|\langle \mathcal{R}_t \mathrm{h}, \psi_z^{\lambda_0}\rangle\Big|}{t^{\frac{\tilde\eta}{2}}\lambda^\alpha e_{\ell_0+t}(z) p_{\delta+2a}(z)}\bigg)^2 dz\bigg)^{1/2} \;.$$

At this point, we write $\langle \mathcal{R}\mathrm{h} = (\langle \mathcal{R}_t \mathrm{h} - \Pi_z \mathrm{h}(z)) + \Pi_z \mathrm{h}(z)$ and argue separately for the two terms. The reconstruction bound (III.21) yields a term of order $\|\mathrm{h}\|_{\tilde\gamma,\tilde\eta,\ell_0,\delta,t}\lambda^{-\alpha}$ while the other terms yields a sum over ζ of

$$\begin{aligned}\bigg(\int_{z\in\mathbb{R}^3} \sup_{\psi\in\mathcal{B}_r}\bigg(\frac{\Big|\langle \Pi_z \mathrm{h}(z), \psi_z^{\lambda_0}\rangle\Big|}{t^{\frac{\tilde\eta}{2}}\lambda^\alpha e_{\ell_0+t}(z) p_{\delta+2a}(z)}\bigg)^2 dz\bigg)^{1/2} &\lesssim \|\mathrm{h}\|_{\tilde\gamma,\tilde\eta,\ell_0,\delta,t}\bigg(\frac{t^{\frac{\tilde\eta-\zeta}{2}}\lambda_0^\zeta}{t^{\frac{\tilde\eta}{2}}\lambda^\alpha}\bigg)\\ &\lesssim \|\mathrm{h}\|_{\tilde\gamma,\tilde\eta,\ell_0,\delta,t}\lambda^{-\alpha} \;,\end{aligned}$$

uniformly over all $\lambda \in [\sqrt{t/2}, 1]$. □

Let us write $\mathrm{f} := \mathrm{u}\Xi$ and recall from [HL18, Section 4.1] that $\mathrm{f} \in \mathcal{D}_{w,T}^{\gamma',\eta',2}$ where $\gamma' := \gamma + \alpha$, $\eta' := \eta + \alpha < \alpha$ and the associated norm ensures that for all $\zeta \in \mathcal{A}_{<\gamma'}$ the following quantities are finite

$$\sup_{t\in(0,T]} \Big\| \frac{|\mathrm{f}(t,x)|_\zeta}{t^{\frac{\eta'-\zeta}{2}} w_t^{(1)}(x,\zeta)} \Big\|_{L^2} \tag{III.22}$$

$$\sup_{\lambda\in(0,2]} \sup_{t\in(2\lambda^2,T]} \Big\| \int_{y\in B(x,\lambda)} \lambda^{-3} \frac{|\mathrm{f}(t,y) - \Gamma_{y,x}\mathrm{f}(t,x)|_\zeta}{t^{\frac{\eta'-\gamma'}{2}} w_t^{(2)}(x,\zeta)\lambda^{\gamma'-\zeta}} dy \Big\|_{L^2} \tag{III.23}$$

$$\sup_{t\in(0,T]} \sup_{s\in(t/2,t)} \Big\| \frac{|\mathrm{f}(t,x) - \mathrm{f}(s,x)|_\zeta}{t^{\frac{\eta'-\gamma'}{2}} w_t^{(1)}(x,\zeta)|t-s|^{\frac{\gamma'-\zeta}{2}}} \Big\|_{L^2} \tag{III.24}$$

where the values of the weights are provided in [HL18, Eq 4.1]. Their precise expressions are unimportant for this proof, we only need that there exists some constants $c, C > 0$ such that for all t, x and all ζ, $i \in \{1,2\}$

$$(1 + p_a(x)) w_t^{(i)}(x,\zeta) \le C e_{\ell_0+t}(x) p_c(x) \;,$$

and that c can be taken as small as desired by diminishing κ.
We now apply Proposition III.23 to $\mathrm{f}(t,\cdot)$ with $\tilde\gamma = \gamma'$ and $\tilde\eta = \eta'$:

$$\|\mathcal{R}_t(\mathrm{f}(t,\cdot))\|_{\mathcal{B}^\alpha_{\ell_0+t,c+2a}} \lesssim t^{\frac{\eta'}{2}} ,$$

uniformly over all $t \in (0,T)$. A straightforward adaptation shows the continuity with respect to (q,f).
We turn to the continuity in time. We set $\tilde\gamma = \gamma' - \kappa/2 > 0$ and $\tilde\eta = \eta' - \kappa/2 > -2$. We consider $s \in (t/2,t)$ and we aim at bounding

$$\mathcal{R}_t(\mathrm{f}(t,\cdot)) - \mathcal{R}_s(\mathrm{f}(s,\cdot)) = \mathcal{R}_t(\mathrm{f}(t,\cdot)) - \mathcal{R}_t(\mathrm{f}(s,\cdot)) = \mathcal{R}_t(\mathrm{f}(t,\cdot) - \mathrm{f}(s,\cdot)) .$$

To that end, we establish a bound of the form $\|\mathrm{f}(t,\cdot)-\mathrm{f}(s,\cdot)\|_{\tilde\gamma,\tilde\eta,\ell_0,c,t} \lesssim |t-s|^{\kappa/4}$ uniformly over all $s \in (t/2,t)$. From (III.24), we deduce that

$$\begin{aligned}\sum_{\zeta\in\mathcal{A}_{<\tilde\gamma}} \Big\| \frac{|\mathrm{f}(t,x)-\mathrm{f}(s,x)|_\zeta}{e_{\ell_0+t}(x)p_c(x)} \Big\|_{L^2} &\lesssim t^{\frac{\eta'-\gamma'}{2}}|t-s|^{\frac{\gamma'-\zeta}{2}} = t^{\frac{\tilde\eta-\tilde\gamma}{2}}|t-s|^{\frac{\gamma'-\tilde\gamma}{2}}|t-s|^{\frac{\tilde\gamma-\zeta}{2}}\\ &\lesssim t^{\frac{\tilde\eta-\zeta}{2}}|t-s|^{\frac{\kappa}{4}} .\end{aligned}$$

Regarding the regularity in space, we distinguish two cases. First if $\lambda \le \sqrt{t-s}$ then we bound separately the contributions coming from $\mathrm{f}(t,\cdot)$ and $\mathrm{f}(s,\cdot)$

$$\begin{aligned}&\Big\| \int_{y\in B(x,\lambda)} \lambda^{-3}\frac{|\mathrm{f}(t,y)-\mathrm{f}(s,y)-\Gamma_{y,x}\mathrm{f}(t,x)+\Gamma_{y,x}\mathrm{f}(s,x)|_\zeta}{e_{\ell_0+t}(x)p_c(x)\lambda^{\tilde\gamma-\zeta}}dy \Big\|_{L^2(\mathbb{R}^3)}\\ &\lesssim \Big\| \int_{y\in B(x,\lambda)} \lambda^{-3}\frac{|\mathrm{f}(t,y)-\Gamma_{y,x}\mathrm{f}(t,x)|_\zeta}{e_{\ell_0+t}(x)p_c(x)\lambda^{\tilde\gamma-\zeta}}dy \Big\|_{L^2(\mathbb{R}^3)}\\ &+ \Big\| \int_{y\in B(x,\lambda)} \lambda^{-3}\frac{|\mathrm{f}(s,y)-\Gamma_{y,x}\mathrm{f}(s,x)|_\zeta}{e_{\ell_0+t}(x)p_c(x)\lambda^{\tilde\gamma-\zeta}}dy \Big\|_{L^2(\mathbb{R}^3)}\end{aligned}$$

From (III.23) we get a bound of order

$$\lambda^{\gamma'-\tilde\gamma}t^{\frac{\eta'-\gamma'}{2}} \le (t-s)^{\frac{\kappa}{4}}t^{\frac{\tilde\eta-\tilde\gamma}{2}} ,$$

uniformly over all $\lambda \le \sqrt{t-s} \wedge \sqrt{t/2}$, all $s \in (t/2,t)$ and all $t \in (0,T)$. Second if $\sqrt{t-s} \le \lambda \le \sqrt{t/2}$ then we write

$$\begin{aligned}&\Big\| \int_{y\in B(x,\lambda)} \lambda^{-3}\frac{|\mathrm{f}(t,y)-\mathrm{f}(s,y)-\Gamma_{y,x}\mathrm{f}(t,x)+\Gamma_{y,x}\mathrm{f}(s,x)|_\zeta}{e_{\ell_0+t}(x)p_c(x)\lambda^{\tilde\gamma-\zeta}}dy \Big\|_{L^2(\mathbb{R}^3)}\\ &\le \Big\| \int_{y\in B(x,\lambda)} \lambda^{-3}\frac{|\mathrm{f}(t,y)-\mathrm{f}(s,y)|_\zeta}{e_{\ell_0+t}(x)p_c(x)\lambda^{\tilde\gamma-\zeta}}dy \Big\|_{L^2(\mathbb{R}^3)}\\ &+ \Big\| \int_{y\in B(x,\lambda)} \lambda^{-3}\frac{|\Gamma_{y,x}\mathrm{f}(t,x)-\Gamma_{y,x}\mathrm{f}(s,x)|_\zeta}{e_{\ell_0+t}(x)p_c(x)\lambda^{\tilde\gamma-\zeta}}dy \Big\|_{L^2(\mathbb{R}^3)}\end{aligned}$$

The first term can be bounded using (III.24) and yields a bound of order

$$\frac{1}{\lambda^{\tilde\gamma-\zeta}}|t-s|^{\frac{\gamma'-\zeta}{2}}t^{\frac{\eta'-\gamma'}{2}} \le |t-s|^{\frac{\gamma'-\tilde\gamma}{2}}t^{\frac{\tilde\eta-\tilde\gamma}{2}} = |t-s|^{\frac{\kappa}{4}}t^{\frac{\tilde\eta-\tilde\gamma}{2}}\;,$$

uniformly over all $\sqrt{t-s}\le\lambda\le\sqrt{t/2}$, all $s\in(t/2,t)$ and all $t\in(0,T)$. Regarding the second term, we use the bound

$$|\Gamma_{y,x}\tau|_\zeta \lesssim \sum_{\beta\ge\zeta} p_a(x)|y-x|^{\beta-\zeta}|\tau|_\beta \le \sum_{\beta\ge\zeta} p_a(x)\lambda^{\beta-\zeta}|\tau|_\beta\;,$$

together with (III.24) to bound the second term by the sum over all $\beta\ge\zeta$ of

$$\begin{aligned}\Big\| \int_{y\in B(x,\lambda)} \lambda^{-3}\frac{p_a(x)|\mathrm{f}(t,x)-\mathrm{f}(s,x)|_\beta}{e_{\ell_0+t}(x)p_c(x)\lambda^{\tilde\gamma-\beta}}dy\Big\|_{L^2} &\lesssim \frac{(t-s)^{\frac{\gamma'-\beta}{2}}}{\lambda^{\tilde\gamma-\beta}}t^{\frac{\eta'-\gamma'}{2}}\\ &\le (t-s)^{\frac{\kappa}{4}}t^{\frac{\tilde\eta-\tilde\gamma}{2}}\;,\end{aligned}$$

uniformly over all $\sqrt{t-s}\le\lambda\le\sqrt{t/2}$, all $s\in(t/2,t)$ and all $t\in(0,T)$.
To conclude the proof we observe that $t\mapsto\mathcal{R}_t(f(t,\cdot))$ satisfies the reconstruction bound of [HL18, Eq (3.12) of Theorem 3.10] and therefore coincides with $\mathcal{R}(f)$.

III.4.3 The self-adjoint operator

Definition III.24. Fix $q\in\mathscr{M}$. For $t>0$, define the domain $\mathcal{D}_t := L^2_{e_{-t}}\subset L^2$ and the operator

$$P_t : \begin{cases} \mathcal{D}_t & \to \quad L^2\\ f & \mapsto \quad u^{q,f}(t)\end{cases}$$

Define also $P_0 = I$ on L^2.

Proposition III.25. *Fix $q\in\mathscr{M}$. The collection $(P_t)_{t\in[0,T]}$ is a semigroup of operators on $\mathfrak{h}=L^2(\mathbb{R}^3,dx)$ and is strongly continuous with respect to t. In particular, it satisfies the* (Non-increasing domains with dense union)*,* (Semigroup)*,* (Symmetry) *and* (Weak continuity) *properties of Theorem III.3.*

Proof. Non-increasing domains with dense union: It is immediate that $\mathcal{D}_t\subset\mathcal{D}_s$ whenever $s\le t$. Furthermore since $C_c\subset\mathcal{D}_t$ the density property holds. *Semigroup*: This is a consequence of items 1 and 3 of Theorem III.22. *Weak-continuity*: This is a consequence of item 2 of Theorem III.22 that even shows strong continuity in time. *Symmetry*: This is a consequence of item 4 of Theorem III.22 and of the continuity stated in item 1 of Theorem III.22. □

Definition III.26. For $q\in\mathscr{M}$, let $H(q)$ be the unique self-adjoint operator given by Theorem III.3 associated to the symmetric semigroup $(P_t)_{t\in[0,T]}$ of Definition III.7. In particular, P_t is the restriction of $e^{-tH(q)}$ to $\mathcal{D}_t$ for all $t\in[0,T]$.

Remark III.27. Since $\mathcal{D}_t$ is dense in L^2, the proof of [KL81, Lemma 6] shows that $H(q)$ is essentially self-adjoint over $\hat{\mathcal{D}}(q) := \bigcup_{0<s<t} P_s \mathcal{D}_t$.

We now establish a few properties satisfied by $H(q)$. Given any $f \in L^2$, denote by $\mu_f(q)$ the spectral measure associated to the self-adjoint operator $H(q)$ and f, that is, the unique finite measure on $\mathbb{R}$ with Stieltjes transform

$$\int_{\mathbb{R}} (\lambda - z)^{-1} \mu_f(q)(d\lambda) = \langle f, (H(q) - z)^{-1} f \rangle \,, \quad \forall z \in \mathbb{C} \backslash \mathbb{R} \,.$$

Proposition III.28. *1. For any $f \in L^2$, the map $q \mapsto \mu_f(q)$ is continuous from $\mathscr{M}$ into the space of finite non-negative measures endowed with the topology of weak convergence. As a consequence $q \mapsto H(q)$ is continuous in the strong resolvent sense.*

2. When $q = Q_\varepsilon(\xi)$, $H(q)$ coincides with the essentially self-adjoint operator $-\Delta + \xi_\varepsilon + C_\varepsilon$.

Proof. Fix $f \in C_c$. Then $f \in \mathcal{D}_t(q)$ for any $q \in \mathscr{M}$ and any $t \geq 0$. In addition, we have

$$\int_{\mathbb{R}} e^{-s\lambda} \mu_f(q)(\mathrm{d}\lambda) = \left\langle f, e^{-sH(q)} f \right\rangle = \left\langle f, u^{q,f}(s) \right\rangle, \quad \forall s \in [0, t] \,.$$

Suppose (q_n) is a sequence in $\mathscr{M}$ converging to q. Since f has compact support, we deduce from item 1 of Theorem III.22 that

$$\left\langle f, u^{q_n,f}(s) \right\rangle \to \left\langle f, u^{q,f}(s) \right\rangle \,, \quad n \to \infty \,.$$

Therefore the Laplace transform of the measure $\mu_f(q_n)$ converges to the Laplace transform of $\mu_f(q)$. By Lemma III.14, this implies weak convergence of the measures. The rest of the proof is identical to the proof of Proposition III.12. □

We have all the ingredients at hand to define the Anderson Hamiltonian with white noise potential on $\mathbb{R}^3$. For any $\xi \in \Omega$, recall that $Q_\varepsilon(\xi), Q(\xi) \in \mathscr{M}$ and set

$$\mathcal{H}_\varepsilon(\xi) := H(Q_\varepsilon(\xi)) \,, \quad \mathcal{H}(\xi) := H(Q(\xi)) \,,$$

where H is the deterministic map introduced in Definition III.26.

Proof of Theorem III.1 in dimension 3. By Proposition III.28, for any $\xi \in \Omega$ we have $\mathcal{H}_\varepsilon(\xi) = -\Delta + \xi_\varepsilon + C_\varepsilon$. By Lemma III.21, Q is the limit in probability of Q_ε so that Proposition III.28 entails that $\mathcal{H}_\varepsilon$ converges in the strong resolvent sense to $\mathcal{H}$ in probability. Finally, Definition III.26 ensures that, for any $t \geq 0$, the domain of the operator $e^{-t\mathcal{H}}$ contains the set $\mathcal{D}_t$ so in particular all functions $f \in L^2(\mathbb{R}^d, dx)$ with compact support, and $e^{-t\mathcal{H}} f$ coincides with the solution of (III.1) at time t, starting from f at time 0. □

III.4.4 Inclusion of the supports

Let $\mathtt{supp}(Q_\varepsilon)$ and $\mathtt{supp}(Q)$ be the topological supports of the laws of the r.v. Q_ε and Q. The following result is due to Hairer and Schönbauer [HS21].

Theorem III.29. *For any given $\varepsilon \in (0,1)$, it holds $\mathtt{supp}(Q_\varepsilon) \subset \mathtt{supp}(Q)$.*

Proof. We aim at proving that

$$\mathtt{supp}(Q) = \overline{\{\mathscr{R}^{\bar{c}} q_{\mathrm{can}}(h) : \bar{c} \in \mathbb{R}^3, \quad h \in C^\infty_{p_b}\}}^{\mathscr{M}} \ . \tag{III.25}$$

Recall that for all $\xi \in \Omega$, $Q_\varepsilon(\xi) = \mathscr{R}^{\bar{c}_\varepsilon} q_{\mathrm{can}}(\xi_\varepsilon)$ with $\bar{c}_\varepsilon = (c_\varepsilon, c_\varepsilon^{(1,1)}, c_\varepsilon^{(1,2)})$ and $\xi_\varepsilon \in C^\infty_{p_b}$. As a consequence $\mathtt{supp}(Q_\varepsilon)$ belongs to the set on the r.h.s. of (III.25). Consequently, the statement of the theorem is proven once (III.25) is established. To prove this identity, we rely on three ingredients:

1. There exists a shift operator $(h,q) \mapsto T_h q$ which is continuous from $C^\infty_{p_b} \times \mathscr{M}$ into $\mathscr{M}$ and which satisfies

   $$\mathscr{R}^{\bar{c}} q_{\mathrm{can}}(f+h) = T_h \mathscr{R}^{\bar{c}} q_{\mathrm{can}}(f) = \mathscr{R}^{\bar{c}} T_h q_{\mathrm{can}}(f) \ ,$$

   for all $\bar{c} \in \mathbb{R}^3$ and all $f, h \in C^\infty_{p_b}$.
2. If $q \in \mathtt{supp}(Q)$ then $T_h q \in \mathtt{supp}(Q)$ for any $h \in C^\infty_{p_b}$.
3. For any $\bar{c} \in \mathbb{R}^3$, $\mathscr{R}^{\bar{c}} q(0) \in \mathtt{supp}(Q)$.

With these three ingredients at hand, we deduce that for any $\bar{c} \in \mathbb{R}^3$ and any $h \in C^\infty_{p_b}$

$$\mathscr{R}^{\bar{c}} q_{\mathrm{can}}(h) = T_h \mathscr{R}^{\bar{c}} q_{\mathrm{can}}(0) \in \mathtt{supp}(Q) \ ,$$

and since $\mathtt{supp}(Q)$ is a closed set, (III.25) follows.
Let us provide arguments for the three ingredients above. Item 1. is an extension to a weighted setting of [HS21, Theorem 2.4]: the algebraic part works exactly the same while the required analytic bounds in infinite volume are satisfied since we chose the parameter b small enough. Let us prove Item 2. From the continuity of the shift operator and the closedness of the support, it is sufficient to deal with $h \in C^\infty_{p_b} \cap L^2$. Then, the Cameron-Martin theorem ensures that the laws of ξ and $\xi + h$ are equivalent. Furthermore for $\mathbb{P}$-almost all ξ

$$Q(\xi + h) = \lim_\varepsilon Q_\varepsilon(\xi + h) = \lim_\varepsilon T_h Q_\varepsilon(\xi) = T_h Q(\xi) \ .$$

As a consequence the laws of $T_h Q(\xi)$ and $Q(\xi)$ are equivalent and $\mathtt{supp}(Q) = \mathtt{supp}(T_h Q)$. But a general result, see [HS21, Lemma 3.13], shows that $\mathtt{supp}(T_h Q)$ is the closure of $T_h(\mathtt{supp}(Q))$. Consequently if $q \in \mathtt{supp}$, then $T_h q \in \mathtt{supp}(T_h Q) = \mathtt{supp}(Q)$.
Item 3. can be deduced from a general result of Hairer and Schönbauer [HS21]. To apply this result, we need to check that the Assumptions 2-6 from [HS21] are satisfied by the regularity structure for (PAM) in dimension 3: Assumption 2, which is necessary

for the BPHZ theorem, is satisfied ; Assumption 3 is trivially satisfied since we do not consider product of noises nor derivative of noise ; Assumption 4, which requires the integration kernel to be homogeneous, holds. Regarding Assumptions 5 and 6, we have $\mathcal{V} = \mathcal{V}_0 = \{\Xi\mathcal{I}(X_i\Xi)\}$ and the symmetries of the integration kernel imply the desired assumption[3].
In the context of [HS21, Def. 3.3], the annihilator $\mathscr{H}$ is the whole renormalisation group $\mathcal{G}_-$. The desired result is then stated as [HS21, Proposition 3.8]. The strategy of proof, presented in [HS21, Prop 3.20 and 3.21], consists in producing a random shift ζ_δ which is such that $T_{\zeta_\delta}Q(\xi) \to \mathscr{R}^{\bar{c}}q(0)$ in probability as $\delta \downarrow 0$. The shift ζ_δ happens to be a stationnary field living in a finite inhomogeneous Wiener chaos: it is associated, through Itô-Wiener isometry, to compactly supported kernels. The convergence proven in [HS21] thus extends to the weighted setting considered in the present article using the same argument as in (III.7) as we did for the convergence of the renormalised model. □

III.5 IDENTIFICATION OF THE SPECTRUM

In this section, we identify the spectrum of the Anderson Hamiltonian with white noise potential. Our arguments apply simultaneously in dimensions 2 and 3.

For any $x \in \mathbb{R}^d$, let $\mathcal{T}_x$ denote the translation operator, that is, the operator

$$\mathcal{T}_x f(y) := f(y+x) \ , \quad y \in \mathbb{R}^d \ .$$

Each $\mathcal{T}_x$ is unitary (with adjoint $\mathcal{T}_x^* f = \mathcal{T}_x^{-1} f = f(\cdot - x)$) from $L^2(\mathbb{R}^d)$ into itself.

Lemma III.30. *For all $\xi \in \Omega_0$, all $x \in \mathbb{R}^d$ and all $\varepsilon \in (0,1)$*

$$\mathcal{H}_\varepsilon(\theta_x\xi) = \mathcal{T}_x^*\mathcal{H}_\varepsilon(\xi)\mathcal{T}_x \ , \quad \mathcal{H}(\theta_x\xi) = \mathcal{T}_x^*\mathcal{H}(\xi)\mathcal{T}_x \ .$$

Proof. For all $\xi \in \Omega$, we have seen that in both dimensions 2 and 3 the operator $\mathcal{H}_\varepsilon(\theta_x\xi)$ coincides with $-\Delta + \xi_\varepsilon(\cdot - x) + C_\varepsilon$ so that

$$\mathcal{H}_\varepsilon(\theta_x\xi) = \mathcal{T}_x^*\mathcal{H}_\varepsilon(\xi)\mathcal{T}_x.$$

We now restrict to $\xi \in \Omega_0$. In both dimensions, for all $x \in \mathbb{R}^d$

$$\mathcal{H}(\theta_x\xi) = \lim_k \mathcal{H}_{\varepsilon_k}(\theta_x\xi) = \lim_{\varepsilon_k} \mathcal{T}_x^*\mathcal{H}_{\varepsilon_k}(\xi)\mathcal{T}_x = \mathcal{T}_x^*\mathcal{H}(\xi)\mathcal{T}_x$$

where the limits are in the strong resolvent sense and where we have used the fact that $\mathcal{T}_x$ is unitary on L^2. □

[3]The symmetry group introduced in [HS21, Subsection 2.5] and required in the Assumptions 5 and 6 is taken to be the finite group of transformations of $\mathbb{R}^3$ generated by $x \mapsto x$ and $x \mapsto -x$: note that the Green function of the Laplacian is invariant under these tranformations.

We are now in the framework of ergodic random operators. More precisely, it is well-known that $(\theta_x)_{x\in\mathbb{R}^d}$ is an ergodic family of measure-preserving transformations on the probability space $(\Omega,\mathbb{P})$, see for instance [Mat21, Prop. B.1]. For any $z\in\mathbb{C}\backslash\mathbb{R}$, the maps $\xi\mapsto(\mathcal{H}_\varepsilon(\xi)-z)^{-1}$ and $\xi\mapsto(\mathcal{H}(\xi)-z)^{-1}$ are the compositions of the measure maps $\xi\mapsto Q_\varepsilon(xi)$ and $\xi\mapsto Q(\xi)$ with the continuous (see Propositions III.12 and III.28) map $q\mapsto(H(q)-z)^{-1}$: therefore, they are measurable from $(\Omega,\mathbb{P})$ into the set of bounded operators on $L^2(\mathbb{R}^d,dx)$, and we deduce that the random operators $\mathcal{H}_\varepsilon$ and $\mathcal{H}$ are measurable in the sense of [CL90, Definition V.1.3]. These properties, combined with the result of the last lemma, imply that the spectra of $\mathcal{H}_\varepsilon$ and $\mathcal{H}$ are almost surely deterministic sets, see [CL90, Proposition V.2.2]: namely, given $\varepsilon\in(0,1)$, there exist $\Sigma_\varepsilon,\Sigma\subset\mathbb{R}$ such that for $\mathbb{P}$-almost all ξ, the spectrum of $\mathcal{H}_\varepsilon(\xi)$ is Σ_ε and the spectrum of $\mathcal{H}(\xi)$ is Σ. (A priori the sets $\Sigma_\varepsilon,\Sigma$ depend on the dimension at stake but, as we will see, they don't).

Our last ingredient is the identification of the almost sure spectrum of $\mathcal{H}_\varepsilon$: it follows from standard techniques.

Proposition III.31. *For any given $\varepsilon\in(0,1)$ and in dimensions 2 and 3, $\Sigma_\varepsilon=\mathbb{R}$.*

Proof. Fix $r\in\mathbb{R}$. The strategy is to construct a Weyl sequence of $\mathcal{H}_\varepsilon$ for the value r, i.e. a sequence (f_n) in $\mathcal{D}(\mathcal{H}_\varepsilon)$ such that $\|f_n\|_{L^2}=1$ and $\|(\mathcal{H}_\varepsilon-r)f_n\|_{L^2}\to0$. The existence of such a sequence implies that $r\in\Sigma_\varepsilon$ see [Tes14, Lemma 2.17]. To do so, we argue by ergodicity that there exist arbitrary large regions of space where ξ_ε is "flat" so that we can use a Weyl sequence of the Laplacian for a well-chosen energy.

For $n\geq1$, define the event

$$E_n:=\left\{\xi\in\Omega\middle|\exists z\in\mathbb{R}^d,\forall y\in\mathbb{R}^d:|y-z|\leq n\Rightarrow|\xi_\varepsilon(y)+C_\varepsilon-r|\leq\frac{1}{n}\right\}\ ,$$

as well as $E=\bigcap_{n\geq1}E_n$. We claim that $\mathbb{P}(E)=1$. First observe that the event E_n is invariant with respect to the shift operators $(\theta_x)_{x\in\mathbb{R}^d}$. The ergodicity of ξ_ε thus implies that each event E_n has probability either 0 or 1. However, the explicit gaussian structure of ξ_ε allows to show that $\mathbb{P}(E_n)\geq\mathbb{P}(\sup_{|y|\leq n}|\xi_\varepsilon(y)+C_\varepsilon-r|\leq\frac{1}{n})>0$. As a consequence $\mathbb{P}(E_n)=1$ and therefore $\mathbb{P}(E)=\mathbb{P}(\bigcap_n E_n)=1$.

Fix $\xi\in E$. For any $n\geq1$, we can find $z_n(\xi)\in\mathbb{R}^d$ such that $\sup_{y:|y-z_n(\xi)|\leq n}|\xi_\varepsilon(y)-r+C_\varepsilon|\leq1/n$. Let $\chi\in C_c^\infty$ be such that $0\leq\chi\leq1$, $\chi\equiv1$ on $B(0,1/2)$ and χ is supported in $B(0,1)$. Set $\chi_n=\chi((\cdot-z_n(\xi))/n)$ and define the function f_n by

$$f_n(x)=\frac{c}{n^{d/2}}\prod_{j=1}^{d}\sin\left(\frac{x_j}{n}\right)\chi_n(x),\quad x\in\mathbb{R}^d,$$

where the constant $c>0$ is chosen so that f_n is normalized in L^2 (it is independent of n). Note that $\|\nabla f_n\|_{L^2}\leq\frac{C}{n}$ and $\|\Delta f_n\|_{L^2}\leq\frac{C}{n^2}$ for some constant $C>0$ independent of n. It is clear that f_n is in the domain of $\mathcal{H}_\varepsilon(\xi)=-\Delta+\xi_\varepsilon+C_\varepsilon$. In addition

$$\|(\mathcal{H}_\varepsilon-r)f_n\|_{L^2}\leq\|\Delta f_n\|_{L^2}+\|(\xi_\varepsilon+C_\varepsilon-r)f_n\|_{L^2}\leq\frac{C}{n^2}+\frac{1}{n}\to0\ .$$

Therefore (f_n) is a Weyl sequence for $\mathcal{H}_\varepsilon(\xi)$ at energy r. Since E has full measure, this proves that $r \in \Sigma_\varepsilon$ and thus $\Sigma_\varepsilon = \mathbb{R}$. $\square$

We now identify the almost sure spectrum of $\mathcal{H}$.

Proof of Theorem III.2. Fix $\varepsilon \in (0, 1)$. There exists a measurable set $\Omega_1 \subset \Omega_0$ of full $\mathbb{P}$-measure on which the spectrum of $\mathcal{H}_\varepsilon(\xi)$ is Σ_ε and the spectrum of $\mathcal{H}(\xi)$ is Σ. Let q be an element in the intersection of $\mathtt{supp}(Q_\varepsilon)$ and $Q_\varepsilon(\Omega_1)$. There exists a sequence q_n that lies in $Q(\Omega_1)$ such that $q_n \to q$ in the space $\mathscr{M}$ as $n \to \infty$. Indeed, if no such sequence exists then there is a ball of radius $\delta > 0$ centered at q which is completely included into $\mathscr{M}\backslash Q(\Omega_1)$. This ball would have zero-measure under the law of Q and therefore q would not lie in $\mathtt{supp}(Q)$: this would raise a contradiction since q lies in $\mathtt{supp}(Q_\varepsilon)$ which is included into $\mathtt{supp}(Q)$ by Theorem III.15 in dimension 2 and Theorem III.29 in dimension 3.

Necessarily there exist $\xi, \xi_n \in \Omega_1$ such that $q = Q_\varepsilon(\xi)$ and $q_n = Q(\xi_n)$. Fix $f \in L^2(\mathbb{R}^d)$ with unit L^2-norm and recall that $\mu_f(q)$, $\mu_f(q_n)$ are the spectral measures associated to $H(q) = \mathcal{H}_\varepsilon(\xi)$ and $H(q_n) = \mathcal{H}(\xi_n)$. Note they are probability measures on $\mathbb{R}$. Since $\xi \in \Omega_1$, the spectrum of $\mathcal{H}_\varepsilon(\xi)$ is Σ_ε. Similarly, the spectrum of $\mathcal{H}(\xi_n)$ is Σ for every $n \geq 1$. By the continuity of the deterministic map H stated in Propositions III.12 and III.28, $\mu_f(q_n)$ converges weakly to $\mu_f(q)$.
Let B be the open set $\mathbb{R}\backslash\Sigma$. Since the support of $\mu_f(q_n)$ is a subset of the spectrum of $\mathcal{H}(\xi_n)$, that is, a subset of Σ, the weak convergence stated above yields

$$\mu_f(q)(B) \leq \liminf_{n\to\infty} \mu_f(q_n)(B) = 0 \; .$$

Consequently, the support of $\mu_f(q)$ is included into Σ. Since the spectrum of $\mathcal{H}_\varepsilon(\xi)$ is the closure of the union over all $f \in L^2$ of the support of $\mu_f(q)$, this suffices to deduce that $\Sigma_\varepsilon \subset \Sigma$. Proposition III.31 shows that $\Sigma_\varepsilon = \mathbb{R}$, and therefore $\Sigma = \mathbb{R}$. $\square$

Chapter IV

Landau Hamiltonian perturbed by Gaussian white noise on $\mathbb{R}^2$ and asymptotic of eigenvalues in finite volume

This chapter is based the draft of an ongoing work.

The statements (Theorem IV.1 and Proposition IV.18) marked by (*) indicate that their proofs are incomplete. The gap lies in Proposition IV.18 where we aim to show that the operator N is essentially self-adjoint over a set $\mathscr{C}$. The proof obtained so far (Proposition IV.17) only allows to deduce N admits a self-adjoint extension, which does not necessarily coincide with its closure. In particular, the domain of this extension could potentially be much larger than the closure of $\mathscr{C}$ by the operator norm of $\mathcal{H}$, therefore insufficient for our purpose of applying Faris-Lavine Theorem (Proposition IV.16).

Let us mention that it is possible to carry out the full space construction as in Chapter III, but it requires the theory of regularity structures. We do not present the details but we intend to implement this in the future.

SUMMARY OF THE PRESENT CHAPTER

IV.1 INTRODUCTION

In this work, we consider the Landau Hamiltonian in dimension 2 perturbed by Gaussian white noise, formally represented by

$$\mathcal{H} := (i\nabla + \mathbf{A})^2 + \xi. \tag{IV.1}$$

In physical context, the vector field $\mathbf{A}$ denotes the magnetic potential, i.e. $\nabla \times \mathbf{A}$ coincides with the external magnetic field. We are interested in the case where the external magnetic field is of constant magnitude $B \geq 0$ and directed along the z-axis. In this case, we choose the *symmetric gauge*, i.e., $\mathbf{A}$ is given by

$$\mathbf{A}(x_1, x_2) = \frac{B}{2}(-x_2, x_1), \quad (x_1, x_2) \in \mathbb{R}^2. \tag{IV.2}$$

The Landau Hamiltonian is an object that appears naturally in various physical models, most notably in that of the quantum Hall effet. Classically, Hall effet is a phenomenon related to the Lorentz force induced by the magnetic field. Indeed, a free charged particle moving in a uniform magnetic field will experience a force perpendicular to the field and to its velocity, and therefore perform a circular motion on an orbit corresponding to the energy of the system. When a constant current of traveling charged particles is confined to a thin conductor sample with applied magnetic field, the Lorentz force is translated to an induced electric field pointed at the direction perpendicular to the current and the applied magnetic field. By measuring the ratio between the magnitudes of the current and of the induced electric potential, one defines the so called Hall conductance $\sigma_{\mathcal{H}}$. In the classical setting, $\sigma_{\mathcal{H}}$ grows linearly with the inverse of the magnetic magnitude $1/B$. However, in the quantum regime, experiments show $\sigma_{\mathcal{H}}$ only takes integer multiples of a fixed quantity, therefore the appellation of quantum Hall effect. More details can be found in the introduction by [BvS94].

Quantum Hall effect is explained by the quantization of the total energy (Hamiltonian) of the system, resulting in the Landau operator, or the magnetic Laplacian

$$(i\nabla + \mathbf{A})^2$$

with $\mathbf{A}$ given by (IV.2). Indeed, on the full space $\mathbb{R}^2$, the spectrum of Landau Hamiltonian is purely essential and composed of eigenvalues given by $(2n+1)B, n = 0, 1, 2, \dots,$ each with infinite multiplicity. These isolated eigenvalues are given the name of *Landau levels*, and are closely related to the quantized growth behaviour of the Hall conductance $\sigma_{\mathcal{H}}$.

A physically relevant consideration is when there are some defects present in the conductor sample. In this case, these defects are usually modeled by some random potential V to capture their random nature, so that the quantization leads to the Hamiltonian of the form

$$(i\nabla + \mathbf{A})^2 + V$$

Then, a natural question is to understand the impact of V on the spectral properties of the Hamiltonian. Firstly, one wants to define the Hamiltonian $(i\nabla + \mathbf{A})^2 + V$ as a self-adjoint unbounded operator on some Hilbert space. Secondly, one studies the spectral nature of such operator, in particular whether the Hamiltonian exhibits spectral localization, i.e. the existence of pure point spectrum in some energy regime with exponentially decaying eigenfunctions.

The most studied form of random potential in the literature is of alloy-type. That is, V takes the form

$$V(x) = \sum_{i\in\mathbb{Z}^d} q_i u(x-i)$$

modeling the defects located at the lattice points of $\mathbb{Z}^d$. In the above expression, u is a deterministic, non-negative, continuous and compactly supported function called the *single-site potential*, modeling the potential introduced by a single defect, whereas (q_i) is an i.i.d. family of random variables named *coupling constants*, each following a probability density g. Most frequently, one puts assumption on the probability density g: When the support g is compact, [CH96] shows that the spectrum of the Hamiltonian has band structure, i.e. the spectrum is contained in the bands of width $2\|V\|_\infty$ centered at the Landau levels, and moreover the spectrum close to band edges is pure point with exponentially decaying eigenfunctions. In the case where the coupling constants, and therefore the potential itself, are unbounded, one expects that the Landau levels are completely smeared out. For instance, in [BCH97], where the probability density g is no longer supposed to be compactly supported, the authors show that under some moment conditions on g the Hamiltonian is self-adjoint with spectrum equals equal to $\mathbb{R}$, and the gaps between Landau levels are partially filled with localised states.

Another choice of V that attracts the physicists' attention is the Gaussian one, i.e. V is a centered Gaussian field with some covariance function $C(x) = \mathbb{E}[V(x)V(0)]$, since this model does not presume a subjacent structure of the defects. Just as in the alloy case with unbounded coupling constants, one expects that the perturbed spectrum is whole $\mathbb{R}$ due to the fact V can take arbitrarily large values with positive probability. Under assumptions on the regularity of the covariance function C, [FLM00] shows this is indeed the case and moreover the spectrum is localised for negative enough energy. In Physics, one also considers the limit when the covariance function $C(x)$ shrinks to a Dirac delta $c\delta(x)$, which corresponds to our case of Gaussian white noise. Although the operator with white noise potential was ill-defined, Wegner [Weg83] already derived an exact expression for the *density of states restricted to a certain Landau level*, a quantity which measures the number of states per unit volume and per infinitesimal energy interval for the Hamiltonian projected to a Landau level.

To our knowledge, there exists no further study on Landau Hamiltonian with the Gaussian white noise in the mathematical community. The current work therefore serves as a first try to bridge the gap between mathematics and physics. For a complete survey on random Schrödinger operators, we refer the readers to [LMW03].

The first challenge one faces in defining $\mathcal{H}$ is the low regularity of the white noise ξ: As ξ is almost surely only a distribution, one immediately remarks that the domain of $\mathcal{H}$ can not contain any smooth functions. On the other hand, if f is to be in the domain of

$\mathcal{H}$, one expects to have some cancellation between $(i\nabla + \mathbf{A})^2 f$ and $\xi \cdot f$ so that the sum can be a L^2-function. As ξ is locally of Hölder regularity α a.s. for any given $\alpha < -d/2$, this requires the domain elements to have regularity $2 + \alpha$. Consequently, one checks that the product $\xi \cdot f$ is only well-defined for dimension $d = 1$ and singular for $d \geq 2$.

Such ill-defined products also constitute a major difficulty for the study of singular SPDEs, to deal with which a procedure called renormalisation is necessary: one first mollifies the noise by an approximation of unity and then removes suitably chosen diverging constants from the regularised solution before taking the limit. It was only recently that major breakthroughs were obtained in this field when Hairer [Hai14] introduced the theory of regularity structures and Gubinelli, Imkeller and Perkowski [GIP15] the paracontrolled distributions. These theories successfully incorporate the idea of renormalisation into the resolution of these ill-posed SPDE problems.

With these novel tools at hand, there has been progress in the construction of random Schrödinger operators involving white noise. For instance, [AC15] defined the Anderson Hamiltonian with Gaussian white noise (i.e. $\mathbf{A} = 0$) on 2-dimensional torus using the theory of paracontrolled calculus. This construction is later generalized to dimension 3 by [GUZ20]. By the theory of regularity structures, a construction of Anderson Hamiltonian in general dimensions $d \leq 3$ with periodic as well as Dirichlet boundary conditions is proposed in [Lab19]. In the case with magnetic field, [MM21] constructed the Landau Hamiltonian without potential but with white noise magnetic field.

In the current work, the main contribution is twofold: firstly, we construct the operator on full space $\mathbb{R}^2$ where the effects of magnetic field on the spectrum are really visible; secondly, we replace the sophisticated SPDE solution theories by a simple conjugation technique inspired by [HL15] which spares us from complicated renormalisation theories. We thus state the main result which yields the well-definedness of Landau Hamiltonian on the full space.

Theorem IV.1 (*)**.** *The unbounded operator $\mathcal{H}$ given by* (IV.1) *is self-adjoint over a dense domain $\mathcal{D}(\mathcal{H})$ contained in $L^2(\mathbb{R}^2)$.*

The chief obstacle for the full space construction is that the spatial white noise ξ is not *globally* Hölder in $\mathbb{R}^2$. To deal with this, the argument here is based on the observation that we can separate the problem of regularity and that of integrability by the decomposition $\xi = \xi^- + \xi^+$, where ξ^- is a distribution living in a global Hölder space while ξ^+ is a unbounded function whose growth can be controlled. Such decomposition is already noted by Gubinelli and Hofmanová [GH19] using Fourier analysis. In section IV.3.1, we shall propose an alternative proof of this fact using wavelet decomposition.

Thanks to the decomposition, we write

$$\mathcal{H} = T + \xi^+, \quad \text{where} \quad T = (i\nabla + \mathbf{A})^2 + \xi^-.$$

This allows us to factor the proof for Theorem IV.1 in two steps.
Firstly, we deal with the irregular part and construct the operator T: instead of using singular SPDE theories, we conjugate T by a stochastic object constructed from the globally Hölder noise ξ^-. In some sense, such stochastic object plays the role of *models*

or *enhanced noise*. The so obtained conjugated operator $\bar{T}$ has a better behaviour in terms of regularity, and we are able to study its associated quadratic form and define the operator itself via Friedrichs extension. We note that this part of construction can be seen as a special case of the recent work of Matsuda and van Zuijlen [Mv22] where they constructed the Anderson Hamiltonian on bounded domain perturbed by a variety of noises in dimension up to 3.
Secondly, T being realised as a self-adjoint operator, the definition of $\mathcal{H} = T + \xi^+$ can be achieved by a classical result due to Faris and Lavine [FL74], which requires us to find an auxillary self-adjoint operator N and control the commutator $[T + \xi^+, N]$. This observation is first remarked by Ugurcan in [Ugu22] where he constructed the Anderson Hamiltonian with Gaussian white noise in full space $\mathbb{R}^2$.

The first step of the above construction can be applied to define the operators $\mathcal{H}_L$ on bounded open cubes $Q_L = (-L/2, L/2)^2$ with Dirichlet boundary condition. In this case, $\mathcal{H}_L$ admits compact resolvents and therefore its spectrum is composed of isolated eigenvalues with finite multiplicity $\lambda_{1,L} \leq \lambda_{2,L} \leq \dots$. As a secondary result, we obtain the almost sure asymptotic of $\lambda_{n,L}$ as L goes to infinity. This result is in line with the results of Anderson Hamiltonian proved in [Cv21].

Theorem IV.2. *The eigenvalues of $\mathcal{H}_L = (i\nabla + \mathbf{A})^2 + \xi$ acting on $L^2(Q_L)$ satisfy*

$$\lambda_{n,L} \sim -C \log L \text{ as } L \to +\infty \quad \text{a.s.},$$

where C is the optimal constant in the Ladyzhenskaya inequality: $\|f\|_{L^4} \leq C \|\nabla f\|_{L^2}^{1/2} \|f\|_{L^2}^{1/2}$ for all $f \in H^1$.

Theorem IV.2 is implied by Theorem IV.28 below by a Borel-Cantelli argument. Here we only establish the proof of Theorem IV.28 in section IV.4 and refer the readers to the proof of [HL22, Thm. 1] for the final step. Although the proof strategy here is exactly the same as [Cv21] and [HL22], the intermediate lemmas become more transparent and concise since our construction does not require machinery from regularity structures or paracontrolled distributions.

The article is organised in the following way: In section IV.2 we introduce the function spaces which will be used in the sequel. In section IV.3, we prove Theorem IV.1 by the arguments outlined previously and also give the construction of operators on bounded boxes. In section IV.4, Theorem IV.2 is proved.

IV.2 PRELIMINARIES

For $r \in \mathbb{N} \cup \{+\infty\}$, let $\mathcal{C}^r(\mathbb{R}^d)$ denote the collection of r-times continuously differentiable real-valued functions on $\mathbb{R}^d$. Let $\mathscr{B}^r(\mathbb{R}^d)$ be the family of $\mathcal{C}^\infty$-functions supported in the unit ball of $\mathbb{R}^d$ and whose $\mathcal{C}^r$-norm is bounded by 1. Let $\mathscr{B}^r_k(\mathbb{R}^d)$ be the collection of elements $\eta \in \mathscr{B}^r$ such that $\int_{\mathbb{R}^d} \eta(x)p(x) = 0$ for all polynomials of degree up to k. For $\eta \in \mathscr{B}^r(\mathbb{R}^d)$, $\lambda \in (0, 1]$ and $x \in \mathbb{R}^d$, we define the rescaled function $\eta_x^\lambda = \lambda^{-d}\eta(\frac{\cdot - x}{\lambda})$ and $\eta_x = \eta(\cdot - x)$.

Definition IV.3 (Weighted Besov space)**.** Given constants $\alpha \in \mathbb{R}\backslash\mathbb{N}$, $r > |\alpha|$, $1 \leq p, q < \infty$, and a positive weight function $w : \mathbb{R}^2 \to \mathbb{R}_+\backslash\{0\}$, we define the following spaces:

- For $\alpha < 0$, the space $\mathcal{B}^\alpha_{p,q}(\mathbb{R}^d, w)$ is defined to be the collection of all distributions f such that

$$\|f\|_{\mathcal{B}^\alpha_{p,q}(w)} := \left[\int_0^1 \left(\int_{\mathbb{R}^d} \left| \sup_{\eta \in \mathscr{B}^r(\mathbb{R}^d)} \frac{\left\langle f, \eta_x^\lambda \right\rangle}{w(x)\lambda^\alpha} \right|^p \mathrm{d}x \right)^{\frac{q}{p}} \frac{\mathrm{d}\lambda}{\lambda}\right]^{\frac{1}{q}} < \infty \tag{IV.3}$$

- For $\alpha \geq 0$, the space $\mathcal{B}^\alpha_{p,q}(\mathbb{R}^d, w)$ is defined to be the collection of all real-valued functions f such that

$$\begin{aligned} \|f\|_{\mathcal{B}^\alpha_{p,q}(w)} := & \left(\int_{\mathbb{R}^d} \sup_{\eta \in \mathscr{B}^r(\mathbb{R}^d)} |\langle f, \eta_x \rangle|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \\ & + \left[\int_0^1 \left(\int_{\mathbb{R}^d} \left| \sup_{\eta \in \mathscr{B}^r_{\lfloor \alpha \rfloor}(\mathbb{R}^d)} \frac{\left\langle f, \eta_x^\lambda \right\rangle}{w(x)\lambda^\alpha} \right|^p \mathrm{d}x \right)^{\frac{q}{p}} \frac{\mathrm{d}\lambda}{\lambda}\right]^{\frac{1}{q}} < \infty \end{aligned} \tag{IV.4}$$

In the case where $p = q = \infty$, $\mathcal{B}^\alpha_{p,q}(\mathbb{R}^d, w)$ coincides with the Hölder space $\mathcal{C}^\alpha(\mathbb{R}^d, w)$:

- For $\alpha < 0$, the space $\mathcal{B}^\alpha_{\infty,\infty}(\mathbb{R}^d, w) = \mathcal{C}^\alpha(\mathbb{R}^d, w)$ is defined to be the collection of all distributions f such that

$$\|f\|_{\mathcal{C}^\alpha(w)} := \sup_{\eta \in \mathscr{B}^r(\mathbb{R}^d)} \sup_{\lambda \in (0,1]} \sup_{x \in \mathbb{R}^d} \left| \frac{\left\langle f, \eta_x^\lambda \right\rangle}{w(x)\lambda^\alpha} \right| < \infty \tag{IV.5}$$

- For $\alpha \geq 0$, the space $\mathcal{B}^\alpha_{\infty,\infty}(\mathbb{R}^d, w) = \mathcal{C}^\alpha(\mathbb{R}^d, w)$ is defined to be the collection of all real-valued functions f such that

$$\|f\|_{\mathcal{C}^\alpha(w)} := \sup_{x \in \mathbb{R}^d} |\langle f, \eta_x \rangle| + \sup_{\eta \in \mathscr{B}^r_{\lfloor \alpha \rfloor}(\mathbb{R}^d)} \sup_{\lambda \in (0,1]} \sup_{x \in \mathbb{R}^d} \left| \frac{\left\langle f, \eta_x^\lambda \right\rangle}{w(x)\lambda^\alpha} \right| < \infty \tag{IV.6}$$

From now on, let us a wavelet basis which will be used without further recall in the sequel. Fix a large $r \in \mathbb{N}$ (in fact, $r \geq 2$) and pick a function $\varphi \in \mathcal{C}^r$ with compact support and unit L^2-norm. There exists a finite set Ψ of compactly supported $\mathcal{C}^r$-functions such that the following set constitutes an orthonormal basis in L^2:

$$\{\varphi_x, \psi_y^n : x \in \Lambda, \psi \in \Psi, n \geq 0, y \in \Lambda_n\}$$

where $\Lambda = \mathbb{Z}^d$, $\Lambda_n := 2^{-n}\Lambda$, $\varphi_x := \varphi(\cdot - x)$, and $\psi_y^n(z) = 2^{nd/2}\psi(2^n(z-y))$. Standard wavelet analysis asserts that (φ_x, ψ_y^n) form a orthonormal basis in $L^2(\mathbb{R}^d)$.

With the wavelet basis, we have the following characterization for Hölder space $\mathcal{C}^\alpha$.

Proposition IV.4. *Let $\alpha \in \mathbb{R}$ such that $|\alpha| < r$. A function f belongs to $\mathcal{C}^\alpha$ if and only if*

$$\sup_{x\in\Lambda} \frac{|\langle f, \varphi_x\rangle|}{w(x)} + \sup_{\psi\in\Psi} \sup_{n\geq 0} \sup_{y\in\Lambda_n} \left| \frac{\left\langle f, \psi_y^n \right\rangle}{w(y) 2^{-n(d/2+\alpha)}} \right| < \infty \tag{IV.7}$$

Definition IV.5 (Weighted magnetic Sobolev space)**.** With $\mathbf{A} : \mathbb{R}^d \to \mathbb{R}^d$ given by (IV.2), $\alpha \in \mathbb{R}$ and a positive weight function $w : \mathbb{R}^d \to \mathbb{R}_+ \backslash \{0\}$, we define the norm of a complex valued function $f : \mathbb{R}^d \to \mathbb{C}$

$$\|f\|^2_{H^1_{\mathbf{A}}(w)} := \left\| \frac{f}{w} \right\|^2_{L^2} + \left\| \frac{(i\nabla + \mathbf{A})f}{w} \right\|^2_{L^2}.$$

The weighted magnetic Sobolev $H^1_{\mathbf{A}}(\mathbb{R}^d, w)$ is the family of functions f such that $\|f\|_{H^1_{\mathbf{A}}(w)} < \infty$. When $w \equiv 1$, we write simply $H^1_{\mathbf{A}}(\mathbb{R}^d)$ or $H^1_{\mathbf{A}}$.

We shall use frequently the following important inequality in the sequel.

Lemma IV.6 (Diamagnetic inequality)**.** *For $f \in L^2_{\text{loc}}(\mathbb{R}^d)$ such that $(i\nabla + \mathbf{A})f \in L^2_{\text{loc}}(\mathbb{R}^d)$, one has $|f| \in H^1_{\text{loc}}(\mathbb{R}^d)$ and*

$$|\nabla |f|| \leq |(i\nabla + \mathbf{A})f|, \quad a.e.$$

Proof. [FH10, Thm. 2.1.1] □

notation

In the article, we will use weights of the form $w(x) = \langle x\rangle^\gamma$, $e^{-2Y(x)}$ or $\langle x\rangle^\gamma e^{-2Y(x)}$, where $\langle x\rangle = (1 + |x|^2)^{1/2}$, $\gamma \in \mathbb{R}$ and the random function Y is defined in section IV.3.2. Note in particular that Y is almost surely bounded.

- For a function $Y \in \mathcal{C}^{1-\kappa}(\mathbb{R}^d)$, $L^2_Y(\mathbb{R}^d)$ denotes the weighted Hilbert space $L^2(\mathbb{R}^d, e^Y)$ equipped with the scalar product $\langle f, g\rangle_{L^2_Y} = \int_{\mathbb{R}^d} f(x)g(x)e^{-2Y(x)}\,\mathrm{d}x$.
- $H^\alpha(\mathbb{R}^d, w)$ denotes the weighted Sobolev space $\mathcal{B}^\alpha_{2,2}(\mathbb{R}^d, w)$.
- On a bounded open set $\Omega \subset \mathbb{R}^d$, $H^\alpha(\Omega)$ denotes the usual Sobolev space with regularity α on Ω. $H^1_0(\Omega)$ denotes the closure of $C^\infty_c(\Omega)$ under $\mathcal{H}^1$-norm. $H^1_{\mathbf{A}}(\Omega)$ denotes the family of functions f defined on Ω such that $\|f\|^2_{H^1_{\mathbf{A}}} := \|f\|^2_{L^2} + \|(i\nabla + \mathbf{A})f\|^2_{L^2} < \infty$.
- Throughout the article, C_ω denotes an a.s. finite constant depending only the realisation of the noise ξ and is independent of everything else. Its value can differ from line to line.

IV.3 CONSTRUCTION OF $\mathcal{H}$ ON $\mathbb{R}^2$

In this section, the objective is to prove Theorem IV.1, i.e. construct the operator **H** on full space.

We first provide a proof of the decomposition of noise using wavelet basis in section IV.3.1. We then elaborate on the construction of operator $T = (i\nabla + \mathbf{A})^2 + \xi^-$ in section IV.3.2. In section IV.3.3, we proceed to treat the unbounded part of noise ξ^+ by Faris-Lavine theorem. Finally, we give a brief remark on the construction over bounded boxes in section IV.3.4.

IV.3.1 DECOMPOSITION OF NOISE

Consider the Gaussian white noise ξ on $\mathbb{R}^d$, $d \geq 1$, which lives in the Hölder space $\mathcal{C}^{-d/2-\kappa}(K)$ for every compact set K in $\mathbb{R}^d$. Let $b_x := \langle \xi, \varphi_x \rangle$ and $a_y^{n,\psi} := \left\langle \xi, \psi_y^n \right\rangle$. Note that $b_x, a_y^{n,\psi}$ are iid standard Gaussian random variables.

Let $a, \kappa > 0$ be arbitrarily small positive constants. Define

$$\xi^- := \sum_{x\in\Lambda} b_x \varphi_x + \sum_{\psi\in\Psi} \sum_{n\geq 0} \sum_{y\in\Lambda_n : |y|\leq \ell_n} a_y^{n,\psi} \psi_y^n, \tag{IV.8}$$

$$\xi^+ := \sum_{x\in\Lambda} b_x \varphi_x + \sum_{\psi\in\Psi} \sum_{n\geq 0} \sum_{y\in\Lambda_n : |y| > \ell_n} a_y^{n,\psi} \psi_y^n, \tag{IV.9}$$

where

$$\ell_n := 2^{n(\frac{d+3\kappa}{2a})}.$$

Note that $\xi = \xi^- + \xi^+$ in the sense of distribution.

Proposition IV.7. *Almost surely, $\xi^- \in \mathcal{C}^{-d/2-\kappa}(\mathbb{R}^d, 1)$ and $\xi^+ \in \mathcal{C}^{\kappa}(\mathbb{R}^d, \langle x\rangle^a)$ almost surely. In particular, we have $\sup_{x\in\mathbb{R}^d} \frac{|\xi^+(x)|}{(1+|x|^2)^{a/2}} < \infty$ a.s.*

Proof. Let us begin by ξ^+ : It suffices to show $\mathbb{P}(\|\xi^+\|_{\mathcal{C}^\kappa(\langle x\rangle^a)} > c) \to 0$ as $c \to \infty$. By Proposition IV.4, one has

$$\|\xi^+\|_{\mathcal{C}^\kappa(\langle x\rangle^a)} \asymp \sup_{y\in\Lambda} \frac{|b_y|}{(1+|y|^2)^{a/2}} + \sup_{\psi\in\Psi} \sup_{n\geq 0} \sup_{y\in\Lambda_n : |y|>\ell_n} \frac{|a_y^{n,\psi}|}{(1+|y|^2)^{a/2} 2^{-n(d/2+\kappa)}}$$

We only consider the second term on the right hand side since the first term is simpler and can be dealt with similarly. Using the Gaussian tail bound $\mathbb{P}(|a_y^{n,\psi}| > x) \leq 2(2\pi)^{-1/2} \exp(-x^2/2)$, we have:

$$\begin{aligned}
&\mathbb{P}\left(\sup_{\psi\in\Psi} \sup_{n\geq 0} \sup_{y\in\Lambda_n : |y|>\ell_n} \frac{|a_y^{n,\psi}|}{(1+|y|^2)^{a/2} 2^{-n(d/2+\kappa)}} > c \right) \\
&= 1 - \prod_{\psi\in\Psi} \prod_{n\geq 0} \prod_{y\in\Lambda_n : y>\ell_n} \left[1 - \mathbb{P}\left(|a_y^{n,\psi}| > c(1+|y|^2)^{a/2} 2^{-n(d/2+\kappa)} \right) \right] \\
&\lesssim 1 - \prod_{\psi\in\Psi} \prod_{n\geq 0} \prod_{y\in\Lambda_n : y>\ell_n} \exp\left\{ \log\left[1 - \exp\left(-\frac{c^2}{2} (1+|y|^2)^a 2^{-n(d+2\kappa)} \right) \right] \right\}
\end{aligned}$$

Note that with our choice of ℓ_n, we have $(1+|y|^2)^a 2^{-n(d+2\kappa)} \geq 1$ for all $n \geq 0$ and $|y| > \ell_n$. Consequently, taking large $c > 0$ allows us to make $\exp(-\frac{1}{2}c^2(1+|y|^2)^a 2^{-n(d+2\kappa)})$ small uniformly for all n. As $\log(1-u) \geq -2u$ for all u small enough, it follows that

$$\mathbb{P}\left(\sup_{\psi\in\Psi}\sup_{n\geq 0}\sup_{y\in\Lambda_n:|y|>\ell_n}\frac{|a_y^{n,\psi}|}{(1+|y|^2)^{a/2}2^{-n(d/2+\kappa)}} > c\right)$$
$$\lesssim 1-\exp\left\{-2\sum_{n\geq 0}\sum_{y\in\Lambda_n:|y|>\ell_n}\exp\left[-\frac{c^2}{2}(1+|y|^2)^a 2^{-n(d+2\kappa)}\right]\right\}$$

for c large enough. The double sum can be estimated by

$$\sum_{n\geq 0}\sum_{y\in\Lambda_n:|y|>\ell_n}\exp\left[-\frac{c^2}{2}(1+|y|^2)^a 2^{-n(d+2\kappa)}\right]$$
$$\asymp \sum_{n\geq 0}2^{nd}\int_{\ell_n}^{\infty}\exp\left[-\frac{c^2}{2}(1+r^2)^a 2^{-n(d+2\kappa)}\right] r^{d-1}\,\mathrm{d}r$$
$$\lesssim \sum_{n\geq 0}2^{nd}\ell_n^{d-1}\exp\left[-\frac{c^2}{2}(1+\ell_n^2)^a 2^{-n(d+2\kappa)}\right] \lesssim \sum_{n\geq 0}2^{nd}\ell_n^{d-1}e^{-2^{n\kappa}c^2/2} \lesssim e^{-c^2/2}.$$

Therefore we have indeed $\lim_{c\to\infty}\mathbb{P}(\|\xi^+\|_{\mathcal{C}^\kappa(\langle x\rangle^a)} > c) \lesssim \lim_{c\to\infty} 1-e^{-2e^{-c^2/2}} = 0$ and thus $\|\xi^+\|_{\mathcal{C}^\kappa(\langle x\rangle^a)} < \infty$ a.s.. In particular, by the embedding $\mathcal{C}^\kappa(\langle x\rangle^a) \hookrightarrow L^\infty(\langle x\rangle^a)$, we have $\sup_{x\in\mathbb{R}^d}\frac{|\xi^+(x)|}{(1+|x|^2)^{a/2}} < \infty$ a.s..

It remains to show ξ^- is a.s. globally Hölder $\alpha := -d/2-\kappa$, i.e.

$$\sup_{y\in\Lambda}|b_y| + \sup_{\psi\in\Psi}\sup_{n\geq 0}\sup_{y\in\Lambda_n:|y|\leq\ell_n}\frac{|a_y^{n,\psi}|}{2^{-nd/2}2^{-n\alpha}} < +\infty \quad \text{a.s.}$$

For $p \geq 1$, we have

$$\mathbb{E}\left[\sup_{\psi\in\Psi}\sup_{n\geq 0}\sup_{y\in\Lambda_n:|y|\leq\ell_n}\left(\frac{|a_y^{n,\psi}|}{2^{-nd/2}2^{-n\alpha}}\right)^{2p}\right] \lesssim \sum_{\psi\in\Psi}\sum_{n\geq 0}\sum_{y\in\Lambda_n:|y|\leq\ell_n}2^{2np(d/2+\alpha)}\,\mathbb{E}[|a_y^{n,\psi}|^2]^p$$
$$\lesssim \sum_{n\geq 0}2^{2np(d/2+\alpha)}(2^n\ell_n)^d = \sum_{n\geq 0}2^{-2np\kappa}2^{nd(1+\frac{d+3\kappa}{2a})}$$

By picking p large enough, the rightmost series is convergent, and therefore one deduces that the triple supremum is finite a.s.. For the other term involving b_y the argument is the same. We conclude that $\xi^- \in \mathcal{C}^\alpha(\mathbb{R}^d)$ a.s.. □

IV.3.2 Construction with globally Hölder noise

We now proceed to define the Landau Hamiltonian with globally Hölder noise

$$T = (i\nabla + \mathbf{A})^2 + \xi^- \tag{IV.10}$$

where $\xi^- \in \mathcal{C}^{-d/2-\kappa}(\mathbb{R}^2)$ is as in Proposition IV.7. The idea of this construction is to utilize the exponential change of variable introduced by [HL15]: Let $G(x) = -\frac{\log|x|}{2\pi}\chi(x)$ with χ being a smooth cutoff function evaluated to 1 in the unit ball $B(0,1)$ and supported in $B(0,2)$. Then, one has $-\Delta G = \delta + F$ for some smooth function F supported outside $B(0,1)$.

Define $Y = G * \xi^-$, which is now a smoother noise and belongs to $\mathcal{C}^{1-\kappa}$ a.s.. Recall the notation $L^2_Y(\mathbb{R}^2) = L^2(\mathbb{R}^2, e^{-2Y(x)}\,\mathrm{d}x)$. Now, instead of contructing $T : \mathcal{D}(T) \subset L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$, we can consider the conjugated operator $\bar{T} : \mathcal{D}(\bar{T}) \subset L^2_Y \to L^2_Y$ formally given by

$$\bar{T} := e^Y T e^{-Y} = (i\nabla + \mathbf{A})^2 - 2i\nabla Y \cdot (i\nabla + \mathbf{A}) - |\nabla Y|^2 - F * \xi^- \tag{IV.11}$$

Indeed, on has the following simple observation

Lemma IV.8. *Suppose Y are globally $\mathcal{C}^{1-\kappa}$-functions. The mapping $U : g \mapsto e^Y g$ is a bijective, unitary transformation from $L^2(\mathbb{R}^2)$ to $L^2_Y(\mathbb{R}^2)$. As a consequence, for two operators T and $\bar{T} = UTU^{-1}$ defined on L^2 and L^2_Y respectively, T is self-adjoint if and only if $\bar{T}$ is.*

Proof. For any $f, g \in L^2(\mathbb{R}^2)$, one has the equality $\langle Uf, Ug\rangle_{L^2_Y} = \langle f, g\rangle_{L^2}$, which proves the unitarity and in particular U is an isometry. To see that it is also surjective, observe that for all $f \in L^2_Y$, $e^{-Y}f$ is in L^2. □

Observe that in the expression for $\bar{T}$, the noise ξ^- disappears and the only ill-defined term is $|\nabla Y|^2$. This ill-defined product however can be easily dealt with by a simple renormalization procedure as in [HL15], which we summarize below: Fix a mollifing sequence[1] $(\rho_\varepsilon)_\varepsilon$ and pose $\xi^-_\varepsilon = \xi^- * \rho_\varepsilon$ as well as $Y_\varepsilon = G * \xi^-_\varepsilon$.

Proposition IV.9. *Define $c_\varepsilon(x) := \mathbb{E}\,|\nabla Y_\varepsilon(x)|^2$, $x \in \mathbb{R}^2$. Then the sequence of random fields $(Z_\varepsilon)_\varepsilon$ defined by $Z_\varepsilon(x) := c_\varepsilon(x) - |\nabla Y_\varepsilon(x)|^2$ can be shown to converge in $\mathcal{C}^{-\kappa}(\mathbb{R}^2)$ and in probability to a random (Schwarz) distribution Z. For every test function $\eta \in C^\infty_c$, the random variable $\langle Z, \eta\rangle$ is a second-order homogeneous Wiener chaos associated to ξ, formally represented by*

$$\langle Z, \eta\rangle := -\iint \left(\int_x \nabla G(z-x)\cdot\nabla G(z'-x)\eta(x)\,\mathrm{d}x\right)\xi(\mathrm{d}z)\xi(\mathrm{d}z').$$

Proof. Similar to [HL15]. The only difference is that our counter-term c_ε is no longer a constant due to the fact that the law of ξ^- is not translational invariant. □

Replacing the ill-defined product $-|\nabla Y|^2$ by the renormalised noise $Z \in \mathcal{C}^{-\kappa}$, we now claim that the operator

$$\bar{T} = (i\nabla + \mathbf{A}))^2 - 2i\nabla Y \cdot (i\nabla + \mathbf{A})) + Z - F * \xi^- \tag{IV.12}$$

[1] We define $\rho_\varepsilon(\cdot) := \varepsilon^{-2}\rho(\cdot/\varepsilon)$ where ρ is any fixed non-negative, even, smooth function with bounded support.

can be defined by Friedrichs' realization (Theorem I.13). Once $(\bar{T}, \mathcal{D}(\bar{T}))$ being constructed, we can reverse the conjugation to define

$$T := e^{-Y}\bar{T}e^{Y}, \quad \mathcal{D}(T) := e^{-Y}\mathcal{D}(\bar{T}). \tag{IV.13}$$

Let us prove the claim which is the main result of this section.

Proposition IV.10. *The operator $\bar{T}$ formly given by* (IV.12) *is well-defined and self-adjoint over a domain $\mathcal{D}(\bar{T})$ dense in the weighted magnetic Sobolev space $H^1_{\mathbf{A}}(\mathbb{R}^2, e^Y)$.*

As a consequence of Lemma IV.8, the operator T given by (IV.13) *is also well-defined and self-adjoint.*

Proof. Consider the quadratic form associated to $\bar{T}$ in the Hilbert space $L^2_Y(\mathbb{R}^2)$:

$$\mathcal{Q}(v) = \left\langle v, \bar{T}v \right\rangle_{L^2_Y} := \|(i\nabla + \mathbf{A})v\|^2_{L^2_Y} + \left\langle (Z - F * \xi^-)v, v \right\rangle_{L^2_Y}.$$

For convenience, we shall write $\zeta = Z - F * \xi^-$ in what follows. Note that $\mathcal{Q}$ is a well-defined quadratic form on $H^1_{\mathbf{A}}(e^{-Y})$: Given $v \in H^1_{\mathbf{A}}(e^{-Y})$, the first term in $\mathcal{Q}$ is well-defined and finite. For the second term, notice $\langle \zeta v, v \rangle_{L^2_Y} = \langle \zeta |v|, |v| \rangle_{L^2_Y}$ and that $|v|$ is $\mathcal{H}^1(e^{-Y})$ by the diamagnetic inequality (Lemma IV.6) and the fact that both e^{2Y} and e^{-2Y} are almost surely bounded. As $\zeta \in \mathcal{C}^{-\kappa}(\mathbb{R}^2)$, we deduce $\langle \zeta |v|, |v| \rangle_{L^2_Y}$ is well-defined.

Let us show that $\mathcal{Q}$ is semi-bounded from below over $H^1_{\mathbf{A}}(e^{-Y})$. Indeed, we have

$$\mathcal{Q}(v) \geq \|(i\nabla + \mathbf{A})v\|^2_{L^2_Y} - \|\zeta\|_{\mathcal{C}^{-\kappa}} \||v|\|^2_{H^{2\kappa}(e^Y)}, \quad v \in H^1_{\mathbf{A}}(e^{-Y})$$

for any $\kappa \in (0, 1/2)$. Note that our Sobolev space $\mathcal{H}^{2\kappa}(e^Y)$ is weighted by a non-smooth weight e^{-Y}. However, the random function Y is a.s. $(1-\kappa)$-Hölder and thus satisfies $0 < e^{-Y(x_1)} \leq e^{-Y(x_2)} e^{c|x_1 - x_2|^{1-\kappa}}$. We can therefore apply the Sobolev interpolation given in [ST87, sec. 5.1.2]:

$$\||v|\|^2_{H^{2\kappa}(e^{-Y})} \leq C(\|v\|^{1-2\kappa}_{L^2_Y} \||v|\|^{2\kappa}_{H^1(e^{-Y})})^2 \leq \theta C \|\nabla |v|\|^2_{L^2_Y} + C_\theta \|v\|^2_{L^2_Y}$$

for any $\theta \in (0,1)$ and some independent constant C and some constant C_θ depending on the choice of θ. Bounding the noise term by a random uniform constant, and by the diagmagnetic inequality (Lemma IV.6), we shall have

$$\mathcal{Q}(v) \geq (1 - \theta C_\omega)\|(i\nabla + \mathbf{A})v\|^2_{L^2_Y} - C_{\theta,\omega}\|v\|^2_{L^2_Y}$$

for some random constants C_ω and $C_{\theta,\omega}$. For each realisation ω, we choose θ small enough so that $1 - \theta C_\omega \geq 1/2$, resulting in

$$\mathcal{Q}(v) \geq \frac{1}{2}\|(i\nabla + \mathbf{A})v\|^2_{L^2_Y} - C_\omega \|v\|^2_{L^2_Y} \tag{IV.14}$$

In particular, this implies, for any fixed $\lambda > C_\omega$, the quadratic form $\mathcal{Q}(\cdot) + \lambda\|\cdot\|^2_{L^2_Y}$ with domain $H^1_{\mathbf{A}}(e^{-Y})$ is coercive in the Hilbert space L^2_Y. Friedrichs extension therefore assures the existence of a self-adjoint operator $(\bar{T}, \mathcal{D}(\bar{T}))$ associated to the form $\mathcal{Q}$ such that $\mathcal{D}(\bar{T})$ is dense in $H^1_{\mathbf{A}}(e^{-Y})$ with respect to $\mathcal{Q}$. This concludes the proof. □

Corollary IV.11. *The proof of Proposition IV.10 showed that, there exists $\lambda \geq 0$ large enough such that*

$$C_{1,\omega}\|v\|^2_{H^1_{\mathbf{A}}(e^{-Y})} \leq \mathcal{Q}(v) + \lambda\|v\|_{L^2_Y} \leq C_{2,\omega}\|v\|^2_{H^1_{\mathbf{A}}(e^{-Y})} \tag{IV.15}$$

for some almost surely finte positive random constant $C_{1,\omega}$, $C_{2,\omega}$.

Remark IV.12. Lisewise, we define the regularized operators $\bar{T}_\varepsilon := (i\nabla + \mathbf{A}))^2 - 2i\nabla Y_\varepsilon \cdot (i\nabla + \mathbf{A}) + Z_\varepsilon - F * \xi^-_\varepsilon$ and its counterpart T_ε by the same argument in Proposition IV.10, i.e. by considering the Friedrichs extension of the associated quadratic forms $\mathcal{Q}_\varepsilon(v) := \|(i\nabla + \mathbf{A})v\|^2_{L^2_{Y_\varepsilon}} + \langle (Z_\varepsilon - F * \xi^-_\varepsilon)v, v\rangle_{L^2_{Y_\varepsilon}}$. Note that, since the $\mathcal{C}^{-\kappa}$-norm of the random noise $Z_\varepsilon - F * \xi^-_\varepsilon$ is uniformly bounded for ε a.s., (IV.15) holds for $\mathcal{Q}_\varepsilon$ with constants $C_{1,\omega,\varepsilon}$, $C_{2,\omega,\varepsilon}$ uniformly bounded for ε.

The following convergence result relate the operator $(\bar{T}, \mathcal{D}(\bar{T}))$ defined above to the regularized sequence $\bar{T}_\varepsilon$.

Proposition IV.13. *The sequence of regularized operators $\bar{T}_\varepsilon$ converges in strong resolvent sense to $\bar{T}$ as $\varepsilon \to 0$. Thanks to Lemma IV.8, this implies T_ε converges to T in strong resolvent sense.*

Remark IV.14. In fact, we should like to show norm resolvent convergence instead of strong resolvent convergence by noting

$$\left\langle (\bar{T} + \lambda)(u - u_\varepsilon), h \right\rangle_{L^2_Y} = \langle (\zeta_\varepsilon - \zeta)u_\varepsilon, h\rangle_{L^2_Y}, \quad \forall h \in H^1_{\mathbf{A}}(e^Y),$$

where g is a fixed function in L^2_Y with unit norm and $u = (\bar{T} + \lambda)^{-1}g$, $u_\varepsilon = (\bar{T}_\varepsilon + \lambda)^{-1}g$. Here, $\zeta = Z - F * \xi^-$ and $\zeta_\varepsilon = c_\varepsilon - |\nabla Y_\varepsilon|^2 - F * \xi^-_\varepsilon$. By taking $h = u - u_\varepsilon$, the bound (IV.14) implies that the left-hand side of the above equality is bounded from below by $\|u - u_\varepsilon\|_{H^1_{\mathbf{A}}}$ with a proportional constant independent of ε. On the other hand, the righthand side is bounded by

$$\langle (\zeta_\varepsilon - \zeta)u_\varepsilon, u - u_\varepsilon\rangle_{L^2_Y} \lesssim \|\zeta_\varepsilon - \zeta\|_{\mathcal{C}^{-\kappa}} \|u_\varepsilon\|_{H^{2\kappa}} \|u - u_\varepsilon\|_{H^{2\kappa}}.$$

However, we are not able to conclude since the bound

$$\|u - u_\varepsilon\|_{H^{2\kappa}} \lesssim \|u - u_\varepsilon\|_{H^1_{\mathbf{A}}}$$

seems not true in general on full space $\mathbb{R}^2$ due to the presence of unbounded magnetic potential $\mathbf{A}$.

Therefore, we only provide a proof of the weaker notion of strong resolvent convergence.

Proof. Fix λ large enough. We want to show that for all $g \in L^2_Y$, $(\bar{T}_\varepsilon + \lambda)^{-1}g \to (\bar{T} + \lambda)^{-1}g$ in L^2_Y. Note that the resolvent $v_\varepsilon := (\bar{T}_\varepsilon + \lambda)^{-1}g$ (resp. $v := (\bar{T} + \lambda)^{-1}g$) is the unique minimizer in $H^1_{\mathbf{A}}(e^{-Y})$ of the functional $F_\varepsilon(v) = \frac{1}{2}\mathcal{Q}_\varepsilon(v) + \lambda\|v\|^2_{L^2_{Y_\varepsilon}} - \mathrm{Re}\,\langle g, v\rangle_{L^2_{Y_\varepsilon}}$ (resp.

$F(v) = \frac{1}{2}\mathcal{Q}(v) + \lambda\|v\|^2_{L^2_Y} - \mathrm{Re}\,\langle g, v\rangle_{L^2_Y}$). Although a priori, F and F_ε are only defined on $H^1_{\mathbf{A}}(e^{-Y})$ and $H^1_{\mathbf{A}}(e^{-Y_\varepsilon})$, respectively, we can extend them to the unweighted $L^2(\mathbb{R}^2)$ by assigning $F_\varepsilon(v) = \infty$ if $v \notin H^1_{\mathbf{A}}(e^{-Y_\varepsilon})$ and $F(v) = \infty$ if $v \notin H^1_{\mathbf{A}}(e^{-Y})$.

Since e^{-Y_ε} are bounded uniformly for ε a.s., the resolvents $(\bar{T}_\varepsilon + \lambda)^{-1}$ are bounded operators on $L^2(\mathbb{R}^2)$ whose operator norms are bounded uniformly for all ε. Consequently, the minimizers v_ε form a bounded sequence in L^2. We can therefore extract a weakly convergent subsequence (still denoted by v_ε) such that $v_\varepsilon \rightharpoonup h$ for some h in L^2. If we can show that $F(h) \le \liminf_{\varepsilon\to 0} F_\varepsilon(v_\varepsilon)$, then one must have

$$F(h) \le \liminf_{\varepsilon\to 0} F_\varepsilon(v_\varepsilon) \le \liminf_{\varepsilon\to 0} F_\varepsilon(v) = F(v).$$

By uniqueness of the minimizer, one deduces $h = v$ and thus $v_\varepsilon \rightharpoonup v$ in L^2. As the weak resolvent convergence implies the strong one [Tes14, dO09], this would conclude the proof.

Now we are left to show $F(h) \le \liminf_{\varepsilon\to 0} F_\varepsilon(v_\varepsilon)$. As $\lim_\varepsilon \langle g, v_\varepsilon\rangle_{L^2(\mu_\varepsilon)} = \langle g, h\rangle_{L^2_Y}$ and $\|h\|^2_{L^2} \le \liminf_\varepsilon \|v_\varepsilon\|^2_{L^2}$, we only need to demonstrate $\mathcal{Q}(h) \le \liminf_{\varepsilon\to 0} \mathcal{Q}_\varepsilon(v_\varepsilon)$. That is,

$$\int |(i\nabla + \mathbf{A})h|^2 e^{-2Y}\,\mathrm{d}x \le \liminf_{\varepsilon\to 0} \int |(i\nabla + \mathbf{A})v_\varepsilon|^2 e^{-2Y_\varepsilon}\,\mathrm{d}x \tag{IV.16}$$

$$\left\langle e^{-2Y}\zeta h, h\right\rangle_{L^2} = \liminf_{\varepsilon\to 0} \left\langle e^{-2Y_\varepsilon}\zeta_\varepsilon v_\varepsilon, v_\varepsilon\right\rangle_{L^2} \tag{IV.17}$$

where $\zeta = Z - F * \xi^-$ and $\zeta_\varepsilon = Z_\varepsilon - F * \xi^-_\varepsilon$.

For (IV.16), it suffices to note that $e^{-2Y_\varepsilon}(i\nabla + \mathbf{A})v_\varepsilon$ converges weakly to $e^{-2Y}(i\nabla + \mathbf{A})h$ in L^2. Indeed, it is obvious that $(i\nabla + \mathbf{A})v_\varepsilon \rightharpoonup (i\nabla + \mathbf{A})h$ (just by testing against smooth and compactly supported functions); moreover, for all $\varphi \in L^2$, $e^{-Y_\varepsilon}\varphi$ converges strongly to $e^{-Y}\varphi$ in L^2, whence

$$\left\langle e^{-Y_\varepsilon}(i\nabla + \mathbf{A})v_\varepsilon, \varphi\right\rangle_{L^2} = \left\langle (i\nabla + \mathbf{A})v_\varepsilon, e^{-Y_\varepsilon}\varphi\right\rangle_{L^2} \to \left\langle (i\nabla + \mathbf{A})h, e^{-Y}\varphi\right\rangle_{L^2} = \left\langle e^{-Y}(i\nabla + \mathbf{A})h, \varphi\right\rangle_{L^2},$$

i.e. $e^{-Y_\varepsilon}(i\nabla + \mathbf{A})v_\varepsilon \rightharpoonup e^{-Y}(i\nabla + \mathbf{A})h$. (IV.16) then follows immediately.

On the other hand, to show (IV.17), we suppose without loss of generality that $\liminf_{\varepsilon\to 0} \mathcal{Q}_\varepsilon(v_\varepsilon) + \lambda\|v_\varepsilon\|^2_{L^2_Y} < \infty$. Up to subsequences, we have $\sup_\varepsilon \mathcal{Q}_\varepsilon(v_\varepsilon) + \lambda\|v_\varepsilon\|^2_{L^2_{Y_\varepsilon}} < \infty$. Corollary IV.11 then implies $\sup_\varepsilon \|v_\varepsilon\|^2_{H^1_{\mathbf{A}}(e^{-Y_\varepsilon})} < \infty$. In particular, the sequence $|v_\varepsilon|$ is bounded in $\mathcal{H}^1$ by the diamagnetic inequality (Proposition IV.6). By the Besov product rule, we further deduce that v_ε^2 is bounded in the Besov space $\mathcal{B}^1_{1,2}(\mathbb{R}^d)$. Notice we have the embedding:

$$\mathcal{B}^1_{1,2}(\mathbb{R}^d) \subset \mathcal{B}^{1-\kappa}_{1,1+\kappa}(\mathbb{R}^d) \subset \mathcal{B}^{1-\kappa-d\left(1-\frac{1}{1+\kappa}\right)}_{1+\kappa,1+\kappa}(\mathbb{R}^d) = \left(\mathcal{B}^{-s}_{q,q}(\mathbb{R}^d)\right)'$$

where $s = 1 - \kappa - \frac{d\kappa}{1+\kappa} \in [1 - 3\kappa, 1 - 2\kappa]$ and $q = \frac{1+\kappa}{\kappa}$. Note that by choosing $\kappa < \frac{1}{5}$, we have

$$\zeta \in \mathcal{C}^{-\kappa}(\mathbb{R}^d) \subset \mathcal{B}^{-2\kappa}_{\infty,q}(\mathbb{R}^d) \subset \mathcal{B}^{-2\kappa}_{q,q}(\mathbb{R}^d) \subset \mathcal{B}^{-s}_{q,q}(\mathbb{R}^d)$$

As the sequence v_ε^2 is bounded in $(\mathcal{B}_{q,q}^{-s})'$, Banach-Alaoglu theorem says there is a subsequence converging in the weak-* topology $\sigma((\mathcal{B}_{q,q}^{-s})', \mathcal{B}_{q,q}^{-s})$ to some $w \in (\mathcal{B}_{q,q}^{-s})'$. We claim that $h^2 = w$ a.e.. If this is true, then by the fact that $e^{-2Y_\varepsilon}\zeta_\varepsilon$ converges to $e^{-2Y}\zeta$ in $\mathcal{C}^{-\kappa}$ we can infer $h^2 \in (\mathcal{B}_{q,q}^{-s})'$ and that

$$\left\langle h^2, e^{-2Y}\zeta \right\rangle_{((\mathcal{B}_{q,q}^{-s})', \mathcal{B}_{q,q}^{-s})} = \left\langle w, e^{-2Y}\zeta \right\rangle_{((\mathcal{B}_{q,q}^{-s})', \mathcal{B}_{q,q}^{-s})} = \lim_{\varepsilon} \left\langle v_\varepsilon^2, e^{-2Y_\varepsilon}\zeta_\varepsilon \right\rangle_{((\mathcal{B}_{q,q}^{-s})', \mathcal{B}_{q,q}^{-s})}$$

which proves (IV.17). To prove the claim, we use local compactness: fix arbitrarily a compact $K \subset \mathbb{R}^2$ and let K_1 be its 1-fattening. Let χ_K be a smooth function supported in K_1 such that $\chi_K = 1$ on K. Notice that our sequence $v_\varepsilon \chi_K$ is bounded in $\mathcal{H}^1(K_1)$ by the fact that the $H^1_{\mathbf{A}}$-norm is equivalent to $\mathcal{H}^1$-norm in finite volume. The compact embedding $\mathcal{H}^1(K_1) \hookrightarrow L^2(K_1)$ then implies $v_\varepsilon\chi_K \to h\chi_K$ strongly in L^2 and that $v_\varepsilon^2\chi_K^2 \to h^2\chi_K^2$ in L^1. In particular, the uniqueness of weak-$*$ limit shows $h^2 = w$ a.e. on K. As K is chosen arbitrarily, we deduce $h^2 = w$ a.e. on $\mathbb{R}^2$. This terminates the proof. □

IV.3.3 Definition of Landau Hamiltonian on full space

In this section, we turn to the operator $\mathcal{H} = T + \xi^+$ with T being constructed in the previous section. Define

$$\mathscr{C} := e^{-Y}(\mathcal{D}(\bar{T}) \cap H^1_{\mathbf{A}}(\langle x\rangle^{-2} e^Y)). \tag{IV.18}$$

Lemma IV.15. *$\mathscr{C}$ is non-empty and $\mathcal{H}$ is a well-defined symmetric operator over the domain $\mathscr{C}$.*

Proof. To see that $\mathscr{C}$ is non-empty, we use the Babuška-Lax-Milgram theorem [Bab71]: Let U, V be Hilbert spaces and let $B : U \times V \to \mathbb{R}$ be a continuous bilinear form. If

$$\sup_{v\in V:\|v\|_V=1} |B(u,v)| \geq C\|u\|_U, \qquad \sup_{u\in U:\|u\|_U=1} |B(u,v)| > 0, \forall v \in V\backslash\{0\}$$

then for all $f \in V^*$, there exists a unique $u_f \in U$ such that

$$B(u_f, v) = \langle f, v\rangle_{V^*,V}, \quad \forall v \in V$$

For our purpose, we take weighted Hilbert spaces $U = H^1_{\mathbf{A}}(\langle x\rangle^{-2} e^Y), V = \mathcal{H}^1_A(\langle x\rangle^{2} e^Y)$ and let B be the bilinear form obtained by polarizing the quadratic form $\mathcal{Q}(\cdot) + \lambda\|\cdot\|^2_{L^2_Y}$ with λ large enough. By Corollary IV.11, B is well-defined on $U \times V$ and one has

$$B(u,v) \asymp_\omega \langle u, v\rangle_{H^1_{\mathbf{A}}(e^{-Y})} = \langle (i\nabla + \mathbf{A})u, (i\nabla + \mathbf{A})v\rangle_{L^2_Y} + \langle u, v\rangle_{L^2_Y}.$$

B is indeed continuous since

$$\begin{aligned}|B(u,v)| &\lesssim \| \langle x\rangle^2 (i\nabla + \mathbf{A})u\|_{L^2_Y} \| \langle x\rangle^{-2} (i\nabla + \mathbf{A})v\|_{L^2_Y} + \| \langle x\rangle^2 u\|_{L^2_Y} \| \langle x\rangle^{-2} v\|_{L^2_Y} \\ &\leq \|u\|_U \|v\|_V\end{aligned}$$

by Cauchy-Schwarz. Taking $H^1_{\mathbf{A}}(e^Y)$ as the pivot Hilbert space, we note $H^1_{\mathbf{A}}(\langle x\rangle^{-2} e^Y)$ and $H^1_{\mathbf{A}}(\langle x\rangle^{2} e^Y)$ are dual to each other; therefore

$$\sup_{v\in V:\|v\|_V=1} |B(u,v)| \gtrsim \sup_{\|v\|_{H^1_{\mathbf{A}}(\langle x\rangle^2 e^Y)}=1} \langle u,v\rangle_{H^1_{\mathbf{A}}(e^Y)} = \|u\|_{H^1_{\mathbf{A}}(\langle x\rangle^{-2} e^Y)}$$

and similarly

$$\sup_{u\in U:\|u\|_U=1} |B(u,v)| \gtrsim \sup_{\|u\|_{H^1_{\mathbf{A}}(\langle x\rangle^{-2} e^Y)}=1} \langle u,v\rangle_{H^1_{\mathbf{A}}(e^Y)} = \|v\|_{H^1_{\mathbf{A}}(\langle x\rangle^{2} e^Y)}.$$

The hypothesis of Babuška-Lax-Milgram theorem is hence satisfied. Fix $f\in L^2(\langle x\rangle^{-2} e^Y)$, then $V\ni h\mapsto \langle f,h\rangle_{L^2_Y}$ defines a continuous functional on V. Therefore, Babuška-Lax-Milgram theorem implies the existence of a unique representative $u_f\in H^1_{\mathbf{A}}(\langle x\rangle^{2} e^{-Y})$ such that $B(u_f,h)=\langle f,h\rangle_{L^2_Y}$ for all $h\in V$. By construction of $\bar{T}$ (see (I.10)), we conclude u_f lives in the space $\mathcal{D}(\bar{T})\cap H^1_{\mathbf{A}}(\langle x\rangle^{2} e^{-Y})$. Hence $e^{-Y}u_f\in\mathscr{C}$.

It is clear that $\mathcal{H}$ is well-defined and symmetric over $\mathscr{C}$: $\|\mathcal{H}u\|_{L^2}\le \|Tu\|_{L^2}+\|\xi^+\|_{L^\infty(\langle x\rangle^2)}\|u\|_{L^2(\langle x\rangle^{-2})}<\infty$. For $u,v\in\mathscr{C}$, $\langle Hu,v\rangle=\langle Tu,v\rangle+\langle\xi^+u,v\rangle=\langle u,Tv\rangle+\langle u,\xi^+v\rangle=\langle u,Hv\rangle$ due to the fact that T is symmetric over $\mathscr{C}$ and ξ^+ is real. □

The objective is therefore to show H is essentially self-adjoint over $\mathscr{C}$.

We will use the following result due to Faris and Lavine [FL74, Theorem 1], we mention also [Ugu22] where the same argument is utilised to construct the white noise driven Anderson Hamiltonian on the whole space $\mathbb{R}^2$.

Proposition IV.16 (Faris-Lavine). *Let $\mathcal{H}$ be a symmetric operator and N a positive self-adjoint operator with domain $\mathcal{D}(N)\subset\mathcal{D}(\mathcal{H})$. If the commutator $i[\mathcal{H},N]$ satisfies the estimate*

$$\pm i(\langle Nu,\mathcal{H}u\rangle-\langle\mathcal{H}u,Nu\rangle)\le c\,\langle Nu,u\rangle\,,\quad \forall u\in\mathcal{D}(N)$$

i.e. $\pm i[\mathcal{H},N]\le cN$ in the sense of quadratic form, then $\mathcal{H}$ is essentially self-adjoint

We refer to the original paper for the proof. The heuristics behind the claim is the following: Let $u(t)$ be the wave function of a particle evolving according to the Hamiltonian $\mathcal{H}$ (i.e. solution to $i\partial_t u=\mathcal{H}u$) with initial data $u(0)$. What prevents $\mathcal{H}$ from being essentially self-adjoint is that a priori $\mathcal{H}$ is not semi-bounded, and therefore the particle could flee to infinity in finite time, self-adjoint extension would not be possible without specifying a “boundary condition” after the explosion. Such escapade to infinity would not happen if a positive observable associated to u exists (e.g. the distance of the particle to the orgine) and has bounded expected value. Suppose N is a positive observable with $\pm i[\mathcal{H},N]\le cN$. Then $\pm\frac{\mathrm{d}}{\mathrm{d}t}\langle Nu(t),u(t)\rangle=\pm\langle i[\mathcal{H},N]u(t),(t)\rangle\le c\,\langle Nu(t),u(t)\rangle$ and Grönwall's lemma thus yields $\langle Nu(t),u(t)\rangle\le e^{ct}\langle Nu(0),u(0)\rangle$ for all $t>0$, which just says the expected value $\langle N\rangle$ is finite all the time provided that $u(0)$ is sufficiently localised.

Now the goal is to find a positive self-adjoint auxillary operator $(N,\mathcal{D}(N))$ satisfying the hypothesis of Proposition IV.16 for $\mathcal{H}=T+\xi^+$. Note that by Proposition IV.7

ξ^+ is a real-valued random function such that almost surely $|\xi^+(x)| \le C_\omega \langle x \rangle^2$ for all $x \in \mathbf{R}^2$ and an a.s. finite random constant C_ω. In view of this fact, a natural choice for N would be $N = \mathcal{H} + C(1+|x|^2)$ with a large enough C that cancels the growth of ξ^+. Indeed, we have the following

Proposition IV.17. *Take $\mathcal{D}(N) = \mathscr{C}$. Then there exists a random constant $C_\omega < \infty$ a.s. such that for all $C > C_\omega$, the operator $N = \mathcal{H} + C(1+|x|^2)$ defined on $\mathscr{C}$ is symmetric and positive.*

Proof. Write $\langle Nu, u\rangle = \langle Tu, u\rangle + \left\langle (\xi^+ + C\langle x\rangle^2)u, u\right\rangle$. In the sequel, for $u \in \mathcal{D}(T)$, we denote $v = e^Y u \in \mathcal{D}(\bar{T})$. Recall from Lemma IV.8 that the multiplication by e^Y is a unitary transform from L^2 to L^2_Y, (IV.14) thus gives

$$\langle Tu, u\rangle_{L^2} = \left\langle \bar{T}v, v\right\rangle_{L^2_Y} \ge \frac{1}{2}\|(i\nabla + \mathbf{A})v\|^2_{L^2_Y} - C_\omega \|v\|^2_{L^2_Y}$$

for some uniformly bounded random constant C_ω. We have also

$$\left\langle (\xi^+ + C\langle x\rangle^2)u, u\right\rangle_{L^2} = \left\langle (\xi^+ + C\langle x\rangle^2)v, v\right\rangle_{L^2_Y} \ge (C - C_\omega)\left\langle \langle x\rangle^2 v, v\right\rangle_{L^2_Y}.$$

Summing the two inequalities and choose $C > 2C_\omega$, we can deduce

$$\langle Nu, u\rangle \ge \frac{1}{2}\|(i\nabla + \mathbf{A})v\|^2_{L^2_Y} + (C - 2C_\omega)\left\langle \langle x\rangle^2 v, v\right\rangle_{L^2_Y} \ge (C - 2C_\omega)\|v\|^2_{L^2_Y}, \quad \text{(IV.19)}$$

which concludes the proof. □

Proposition IV.18 (*)**.** *We claim that N is essentially self-adjoint over $\mathscr{C}$ and one has*

$$\langle \mathcal{H}u, u\rangle \lesssim \langle Nu, u\rangle, \quad \forall u \in \mathscr{C}.$$

Remark IV.19. We are unable to prove Proposition IV.18 so far: the previous Proposition IV.17 only serves to show that $(N, \mathscr{C})$ admits a self-adjoint extension $(N, \mathcal{D}(N))$. However, this extension does not necessarily coincide with its closure on $\mathscr{C}$. In particular, we cannot promise the closure of $\mathcal{H}$ on $\mathscr{C}$ is contained in $\mathcal{D}(N)$, therefore insufficient for our purpose of applying Faris-Lavine Theorem (Proposition IV.16).

A possible way to go could be to introduce the new reference operator $\mathcal{H}_0 = (i\nabla + \mathbf{A})^2 + C|x|^2$ so that $N = \mathcal{H}_0 + \xi + C$. Since the potential $V(x) = C|x|^2$ is highly confining, $\mathcal{H}_0$ has purely discrete spectrum (at least in the case where $\mathbf{A} = 0$), which might allow to deduce the essential self-adjointness of N over $\mathscr{C}$.

Proposition IV.20. *For all $u \in \mathscr{C}$, one has*

$$\pm \langle i[\mathcal{H}, N]u, u\rangle_{L^2} \lesssim \langle Nu, u\rangle_{L^2}$$

Proof. As multiplication operators commute, we see easily that $[\mathcal{H}, N] = C[T, |x|^2]$. By expanding in quadratic form and passing to L^2_Y by Lemma IV.8, one has

$$\left\langle [T, |x|^2]u, u\right\rangle_{L^2} = \left\langle (i\nabla + \mathbf{A})(|x|^2 v), (i\nabla + \mathbf{A})v\right\rangle_{L^2_Y} - \left\langle (i\nabla + \mathbf{A})v, (i\nabla + \mathbf{A})(|x|^2 v)\right\rangle_{L^2_Y}$$

where $v = e^Y u$. Since $u \in \mathscr{C}$, v is in $H^1_A(\langle x\rangle^2 e^{-Y})$ and $(i\nabla + \mathbf{A})(|x|^2 v) = |x|^2(i\nabla + \mathbf{A})v + 2ixv \in L^2_Y$, whence

$$\left\langle i[T, |x|^2]u, u\right\rangle_{L^2} = \langle 2xv, (i\nabla + \mathbf{A})v\rangle_{L^2_Y} + \langle (i\nabla + \mathbf{A})v, 2xv\rangle_{L^2_Y} = 4\operatorname{Re}\langle xv, (i\nabla + \mathbf{A})v\rangle_{L^2_Y}$$

Using the fact that $\pm 2\operatorname{Re}\langle xv, (i\nabla + \mathbf{A})v\rangle_{L^2_Y} \le \langle |x|^2 v, v\rangle_{L^2_Y} + \|(i\nabla + \mathbf{A})v\|^2_{L^2_Y}$, we deduce from the lower bound (IV.19) that

$$\pm\langle i[\mathcal{H}, N]u, u\rangle_{L^2} \le 2C\left(\left\langle |x|^2 v, v\right\rangle_{L^2_Y} + \|(i\nabla + \mathbf{A})v\|^2_{L^2_Y}\right) \lesssim \langle Nu, u\rangle_{L^2}$$

which concludes the proof. □

Proof of Theorem IV.1. Provided Propositions IV.18 is true, combing Lemma IV.15, IV.20 and IV.16 we deduce that the operator $\mathcal{H}_{\mathbf{A}}$ is essentially self-adjoint over $\mathscr{C}$. □

IV.3.4 Construction over bounded domains

In fact, the contruction prosposed in section IV.3 works equally well for the operator on bounded domain. Indeed, given a bounded open set $\Omega \subset \mathbb{R}^2$, the Gaussian white noise ξ belongs to the class $\mathcal{C}^{-1-\kappa}(\Omega)$ for all $\kappa > 0$ and the decomposition is not necessary. The reasoning in Proposition IV.10 therefore carries over *mutatis mutandis*. Here we briefly sketch the construction of $(i\nabla + \mathbf{A})^2 + \xi$ as an unbounded operator on $L^2(\Omega)$ with Dirichlet boundary condition, denoted by $\mathcal{H}(\Omega, \mathbf{A}, \xi)$ or just $\mathcal{H}_\Omega$ in the sequel.

The principle is still to construct the conjugated operator $\bar{\mathcal{H}}_\Omega = (i\nabla + \mathbf{A})^2 - 2i\nabla Y \cdot (i\nabla + \mathbf{A}) + \zeta$ on the Hilbert space $L^2_Y(\Omega) = L^2_Y(\Omega, e^Y)$, with $Y = G * \xi$ and the renormalised noise $\zeta = \lim(c_\varepsilon - |\nabla Y_\varepsilon|^2 - F * \xi_\varepsilon)$, of which the associated quadratic form is given by

$$\mathcal{Q}_\Omega(v) := \|(i\nabla + \mathbf{A})v\|^2_{L^2_Y(\Omega)} + \langle \zeta v, v\rangle_{L^2_Y(\Omega)}$$

which makes sense for all $v \in C^\infty_c(\Omega)$. Note on bounded domain Ω, the $H^1_{\mathbf{A}}(\Omega)$-norm is equivalent to $H^1(\Omega)$-norm. The argument of Proposition IV.10 shows that there is a constant $C_{\omega,\Omega}$ depending on the realisation of noise and of the domain Ω such that $\mathcal{Q}_\Omega(v) + \lambda\|v\|^2_{L^2_Y(\Omega)} \gtrsim \|v\|^2_{\mathcal{H}^1(\Omega, e^{-Y})}$ for all $\lambda > C_{\omega,\Omega}$. The Friedrichs extension then assures the existence of a self-adjoint operator $(\bar{\mathcal{H}}_\Omega, \mathcal{D}(\bar{\mathcal{H}}_\Omega))$ with a dense domain $\mathcal{D}(\bar{\mathcal{H}}_\Omega)$ in $H^1_0(\Omega)$. In a way analogous to (IV.13), we define the operator $(\mathcal{H}_\Omega, \mathcal{D}(\mathcal{H}_\Omega))$ by

$$\mathcal{H}_\Omega = e^{-Y}\bar{\mathcal{H}}_\Omega e^Y, \quad \mathcal{D}(\mathcal{H}_\Omega) = e^{-Y}\mathcal{D}(\bar{\mathcal{H}}_\Omega).$$

Note that as $H^1_0(\Omega)$ is compactly embedded in $L^2(\Omega)$, the operator $\mathcal{H}$ so defined has compact resolvents. It therefore admits purely discrete spectrum composed of eigenvalues with finite multiplicity:

$$\lambda_1(\Omega, \mathbf{A}, \xi) \le \lambda_2(\Omega, \mathbf{A}, \xi) \le \cdots$$

where $\lambda_n(\Omega, A, \xi)$ denotes the n-th eigenvalue of the operator $\mathcal{H}_\Omega$ in the incresasing order.

In the same line of reasoning, we can construct the regularised operator $\mathcal{H}_{\Omega,\varepsilon} = \mathcal{H}(\Omega, \mathbf{A}, \xi_\varepsilon + c_\varepsilon) := (i\nabla + \mathbf{A})^2 + \xi_\varepsilon + c_\varepsilon$ as well as its conjugated conterpart $\bar{\mathcal{H}}_{\Omega,\varepsilon}$.

The operators constructed on bounded domains have better convergence property as we have additional compacity. We have now norm resolvent convergence as opposed to strong resolvent convergence (Proposition IV.13)

Proposition IV.21. *On bounded domain $\Omega \subset \mathbb{R}^2$, the operators $(\bar{\mathcal{H}}_{\Omega,\varepsilon}, \mathcal{D}(\bar{\mathcal{H}}_{\Omega,\varepsilon}))$ (resp. $(H_{\Omega,\varepsilon}, \mathcal{D}(H_{\Omega,\varepsilon}))$) converge in norm resolvent sense to $(\bar{\mathcal{H}}_\Omega, \mathcal{D}(\bar{\mathcal{H}}_\Omega))$ (resp.$(\mathcal{H}_\Omega, \mathcal{D}(\mathcal{H}_\Omega))$).*

Proof. Fix $\lambda > C_{\omega,\Omega}$ and denote $R^\varepsilon_\lambda = (\bar{\mathcal{H}}_{\Omega,\varepsilon} + \lambda)$, $R_\lambda = (\bar{\mathcal{H}}_\Omega + \lambda)$. Now it suffices to show

$$\lim_{\varepsilon\to 0} \sup_{\|g\|_{L^2}\le 1} \|(R^\varepsilon_\lambda - R_\lambda)g\| = 0.$$

Recall that in Proposition IV.13 we have shown that $R^\varepsilon_\lambda g \to R_\lambda g$ for all $g \in L^2$. In addition, the set

$$\mathcal{A} := \{(R^\varepsilon_\lambda - R_\lambda)g : \varepsilon > 0, g \in L^2, \|g\|_{L^2} \le 1\}$$

is bounded in $H^1_0(\Omega)$ and therefore pre-compact in $L^2(\Omega)$ thanks to the compact embedding $H^1_0(\Omega) \hookrightarrow L^2(\Omega)$. Consequently, for any $\delta > 0$, we can always find a finite set $(g_i)_{i\in I}$ such that $\mathcal{A}$ is covered in L^2 by the union of open balls of radius $\delta > 0$ centered at $(R^\varepsilon_\lambda - R_\lambda)g_i$. Hence we infer

$$\sup_{\|g\|_{L^2}\le 1} \|(R^\varepsilon_\lambda - R_\lambda)g\| \le \delta + \max_{i\in I} \|(R^\varepsilon_\lambda - R_\lambda)g_i\| \xrightarrow[\varepsilon\to 0]{} \delta.$$

The inequality being true for all $\delta > 0$, the assertion is then proved. ☐

Operators	Expression	Domain
$\bar{T}$	$(i\nabla + \mathbf{A})^2 - 2i\nabla Y \cdot (i\nabla + \mathbf{A}) + (\infty - \lvert\nabla Y\rvert^2) - F * \xi^-$	Dense subspace of $H^1_{\mathbf{A}}(e^{-Y})$
$\bar{T}_\varepsilon$	$(i\nabla + \mathbf{A})^2 - 2i\nabla Y_\varepsilon \cdot (i\nabla + \mathbf{A}) + (c_\varepsilon - \lvert\nabla Y_\varepsilon\rvert^2) - F * \xi^-_\varepsilon$	$e^{Y_\varepsilon}\mathcal{D}((i\nabla + \mathbf{A})^2)$
T	$e^Y \bar{T} e^{-Y} = (i\nabla + \mathbf{A})^2 + \xi^- + \infty$	$e^{-Y}\mathcal{D}(\bar{T})$
T_ε	$e^{Y_\varepsilon} \bar{T} e^{-Y_\varepsilon} = (i\nabla + \mathbf{A})^2 + \xi^-_\varepsilon + c_\varepsilon$	$\mathcal{D}((i\nabla + \mathbf{A})^2)$
$\mathcal{H}$	$T + \xi^+$	Closure of $e^{-Y}(\mathcal{D}(\bar{T}) \cap H^1_{\mathbf{A}}(\langle x\rangle^2 e^{-Y})$
$\mathcal{H}_\varepsilon$	$T_\varepsilon + \xi^+$	Closure of $C^\infty_c(\mathbb{R}^2)$
$\bar{\mathcal{H}}$	$e^Y \mathcal{H} e^{-Y} = \bar{T} + \xi^+$	Closure of $\mathcal{D}(\bar{T}) \cap H^1_{\mathbf{A}}(\langle x\rangle^2 e^{-Y})$
$\bar{\mathcal{H}}_\varepsilon$	$e^{Y_\varepsilon} \mathcal{H}_\varepsilon e^{-Y_\varepsilon} = \bar{T}_\varepsilon + \xi^+_\varepsilon$	$e^{Y_\varepsilon}\mathcal{D}(\mathcal{H}_\varepsilon)$

Table IV.1: Summary for the operators constructed

IV.4 EIGENVALUE ASYMPTOTICS ON BOUNDED BOX TOWARD FULL SPACE

Fix $Q_L = (-L/2, L/2)^2$ with $L > 0$ and consider the operator $\mathcal{H}(Q_L, \mathbf{A}, \xi)$ constructed in section IV.3.4. We know $\mathcal{H}(Q_L, \mathbf{A}, \xi)$ admits discrete eigenvalues $\lambda_1(Q_L, \mathbf{A}, \xi) \le \lambda_2(Q_L, \mathbf{A}, \xi) \le \cdots$. In this section, we are interested in the asymptotic as $L \to \infty$ of the smallest eigenvalue $\lambda_1(Q_L, \mathbf{A}, \xi)$.

IV.4.1 INTERMEDIATE PROPERTIES

Proposition IV.22. *1. The spectrum of H is translation invariant in law, i.e. $y \in \mathbb{R}^2$ and bounded open set $\Omega \subset \mathbb{R}^2$,*

$$(\lambda_n(y+\Omega, \mathbf{A}, \xi))_{n\geq 1} = (\lambda_n(\Omega, \mathbf{A}, \xi))_{n\geq 1} \quad \text{in law}$$

2. For $\beta > 0$ and $n \in \mathbb{N}$, one has

$$\beta^2 \lambda_n(Q_L, \mathbf{A}, \xi) = \lambda_n(Q_{L/\beta}, \beta^2 \mathbf{A}, \beta^{2-d/2}\xi) + \delta_\beta \quad \text{in law}$$

where $\delta_\beta = \lim_{\epsilon\to 0}[\beta^2 c_\epsilon(1) - c_{\epsilon/\beta}(\beta^{d/2})]$ and $c_\varepsilon = \mathbb{E}\,|\nabla G * \xi_\varepsilon|^2$. Moreover, $\delta_\beta \to 0$ when $\beta \to 0$.

Proof. 1. Denote by τ_y the operator of translation by y, i.e., to every function u one associates the function $\tau_y u$ given by $\tau_y u(x) = u(y+x)$. One can define $\tau_y \xi$ by the usual extension to distributions. By linearity of $\mathbf{A}$, we see that $\tau_y \mathbf{A}(x) = \mathbf{A}(x+y) = \mathbf{A}(x) + \mathbf{A}(y)$ for all $x, y \in \mathbb{R}^2$. Now observe that for any differentiable function $\phi : \mathbb{R}^2 \to \mathbb{R}$, we have $e^{i\phi}(i\nabla + \mathbf{A})^2 e^{-i\phi} = (i\nabla + \mathbf{A} + \nabla\phi)^2$. In particular, by posing $\phi_y(x_1, x_2) = \frac{B}{2}(-y_2 x_1 + y_1 x_2)$, one has $(i\nabla + \tau_y \mathbf{A})^2 = e^{i\phi_y}(i\nabla + \mathbf{A})^2 e^{-i\phi_y}$, leading to

$$\mathcal{H}(y+\Omega, \mathbf{A}, \xi) = \tau_y^{-1}\mathcal{H}(\Omega, \tau_y\mathbf{A}, \tau_y\xi)\tau_y = \tau_y^{-1} e^{i\phi_y}\mathcal{H}(\Omega, \mathbf{A}, \tau_y\xi)e^{-i\phi_y}\tau_y$$

As the operator $u \mapsto e^{-i\phi_y}\tau_y u$ is unitary, we deduce that $\lambda_n(y+\Omega, A, \xi) = \lambda_n(\Omega, A, \tau_y\xi)$. As the law of ξ is translational invariant, we conclude that $\lambda_n(y+\Omega, A, \xi) = \lambda_n(\Omega, A, \xi)$ in law.

2. Recall that the operator $\mathcal{H}(Q_L, \mathbf{A}, \xi)$ is the norm-resolvent limit of the regularised sequence $\mathcal{H}(Q_L, \mathbf{A}, \xi_\varepsilon + c_\varepsilon(1)) = (i\nabla + \mathbf{A})^2 + \xi_\varepsilon + c_\varepsilon(1)$, where $c_\varepsilon(\beta) = \mathbb{E}[|\nabla G_\varepsilon * (\beta\xi)|^2]$ with $G_\varepsilon = G * \rho_\varepsilon$ (Proposition IV.21). Denote by u_ε the eigenfunction corresponding to the eigenvalue $\lambda_n(Q_L, \mathbf{A}, \xi_\varepsilon + c_\varepsilon(1))$.

For $\beta > 0$, we now consider the rescaled function given by $w_\epsilon(x) = u_\epsilon(\beta x)$ for all $x \in Q_{L/\beta}$ and the rescaled potentials $\beta^{d/2}\mathbf{A}(\beta\cdot)$ and $\beta^d \xi_\epsilon(\beta\cdot)$. Note that $\beta^d\xi_\epsilon(\beta\cdot)$ has same law as $\beta^{d/2}\xi_{\epsilon/\beta}$ and thus its renormalization constant is given by $c_{\epsilon/\beta}(\beta^{d/2})$.

We calculate

$$\begin{aligned}
&\mathcal{H}(Q_{L/\beta}, \beta^{d/2}\mathbf{A}(\beta\cdot), \beta^d\xi_\epsilon(\beta\cdot) + c_{\epsilon/\beta}(\beta^{d/2}))w_\epsilon(x) \\
&= \beta^2[-\Delta + 2i\mathbf{A}(\beta x)\cdot\nabla + (|\mathbf{A}|^2(\beta x) + \xi_\epsilon(\beta x) + c_\epsilon(1))]u_\epsilon(\beta x) + (c_{\epsilon/\beta}(\beta^{d/2}) - \beta^2 c_\epsilon(1))u_\epsilon(\beta x) \\
&= \beta^2\lambda_n(Q_L, \mathbf{A}, \xi_\epsilon + c_\epsilon(1))u_\epsilon(\beta x) + (c_{\epsilon/\beta}(\beta^{d/2}) - \beta^2 c_\epsilon(1))u_\epsilon(\beta x)
\end{aligned}$$

We thus have

$$\beta^2\lambda_n(Q_L, \mathbf{A}, \xi_\epsilon + c_\epsilon(1)) = \lambda_n(Q_{L/\beta}, \beta^{d/2}\mathbf{A}(\beta\cdot), \beta^d\xi_\epsilon(\beta\cdot) + c_{\epsilon/\beta}(\beta^{d/2})) + \beta^2 c_\epsilon(1) - c_{\epsilon/\beta}(\beta^{d/2})$$

Note that $\beta^{d/2}\mathbf{A}(\beta x) = \beta^2\mathbf{A}(x)$. From the norm resolvent convergence the operators (Proposition IV.21) as $\epsilon \downarrow 0$, we deduce that

$$\beta^2\lambda_n(Q_L, \mathbf{A}, \xi) = \lambda_n(Q_{L/\beta}, \beta^2\mathbf{A}, \beta^{2-d/2}\xi) + \delta_\beta,$$

where $\delta_\beta = \lim_{\epsilon\to 0}[\beta^2 c_\epsilon(1) - c_{\epsilon/\beta}(\beta^{d/2})]$. Moreover, a calculation shows

$$\delta_\beta = \beta^2 \lim_{\varepsilon\to 0}\left(\int |\nabla G_\varepsilon|^2 - |\nabla G_{\varepsilon/\beta}|^2\right) \asymp \beta^2 \lim_{\varepsilon\to 0}[\log\varepsilon - \log(\varepsilon/\beta)] = \beta^2\log\beta.$$

In particular, $\lim_{\beta\to 0}\delta_\beta = 0$.

□

Proposition IV.23. *Fix $L \geq 1$ and let $r < L$. Then,*

$$\min_{k\in\mathbb{Z}^2:|k|_\infty<\frac{L}{2r}+\frac{3}{2}} \lambda_1(kr + Q_{\frac{3r}{2}}, \mathbf{A}, \xi) - \frac{K}{r^2} \leq \lambda_1(Q_L, \mathbf{A}, \xi) \leq \min_{k\in\mathbb{Z}^2:|k|_\infty<\frac{L}{2r}} \lambda_1(kr + Q_r, \mathbf{A}, \xi)$$

with a constant K independent of $L, r, \mathbf{A}$ and ξ.

Proof. We first consider the case where the white noise potential ξ is replaced by some continuous bounded potential V. In this case, the upper bound is a simple consequence of the min-max formula for eigenvalues: Let $G_{Q_L}^{\mathbf{A},V}(\varphi) := \int_{Q_L} |(i\nabla + \mathbf{A})\varphi|^2 + V|\varphi|^2$. Then

$$\lambda_1(Q_L, \mathbf{A}, V) = \inf_{\substack{\varphi\in C_c^\infty(Q_L)\\ \|\varphi\|_{L^2}=1}} G_{Q_L}^{\mathbf{A},V}(\varphi) \leq \min_k \inf_{\substack{\varphi\in C_c^\infty(kr+Q_{\frac{3r}{2}})\\ \|\varphi\|_{L^2}=1}} G_{kr+Q_{\frac{3r}{2}}}^{\mathbf{A},V}(\varphi) = \min_{k\in\mathbb{Z}^2:|k|_\infty<\frac{L}{2r}} \lambda_1(kr+Q_r, \mathbf{A}, \xi).$$

For the lower bound, we follow the same argument as in [GK00], which is based on the existence of a specific partition of unity $(\eta_k)_{k\in\mathbb{Z}^2}$ such that each η_k is supported in $Q^{(k)} := kr + Q_{3r/2}$, $\sum_k \eta_k^2 = 1$ and that $\sum_k |\nabla\eta_k(x)|^2 \leq K/r^2$.
The key point is to observe: For every $u \in C_c^\infty(Q_L)$ with unit L^2-norm, define $u_k := u\eta_k$ and we have $(i\nabla + \mathbf{A})u_k = \eta_k(i\nabla + \mathbf{A})u + iu\nabla\eta_k$ and thus

$$\sum_k |(i\nabla + \mathbf{A})u_k|^2 = |(i\nabla + \mathbf{A})u|^2 + \sum_k |\nabla\eta_k|^2|u|^2$$

where the cross term vanishes since $\sum_k \nabla(|\eta_k|^2) = \nabla(\sum_k \eta_k^2) = 0$. Therefore, one deduces

$$\sum_k G_{Q_L}^{\mathbf{A},V}(u_k) = G_{Q_L}^{\mathbf{A},V}(u) + \int_{Q_L}\sum_k |\nabla\eta_k|^2|u|^2\,\mathrm{d}x \leq G_{Q_L}^{\mathbf{A},V}(u) + \frac{K}{r^2}$$

As the form $G_{Q_L}^{\mathbf{A},V}$ is quadratic and $u_k/\|u_k\|$ is supported in $Q^{(k)}$ and of norm 1, the left hand side admits the lower bound:

$$\sum_k G_{Q_L}^{\mathbf{A},V}(u_k) = \sum_k \|u_k\|^2 G_{Q_L}^{\mathbf{A},V}\left(\frac{u_k}{\|u_k\|}\right) \geq \min_k \lambda_1(Q^{(k)}, \mathbf{A}, V)\sum_k \|u_k\|^2 = \min_k \lambda_1(Q^{(k)}, \mathbf{A}, V).$$

Therefore, after taking an infimum over $\{u \in C_c^\infty : \|u\|_{L^2} = 1\}$, one can conclude $\min_k \lambda_1(Q^{(k)}, \mathbf{A}, V) \leq \lambda_1(Q, \mathbf{A}, V) + \frac{K}{r^2}$.

For the white noise ξ case, it suffices to use the bounds for $\lambda_1(Q_L, \mathbf{A}, \xi_\varepsilon + c_\varepsilon(1))$, pass ε to 0 and conclude by the norm-resolvent convergence Proposition IV.21. □

Lemma IV.24. *For any bounded box $Q \subset \mathbb{R}^2$, let $\Phi : \mathcal{C}^{1-\kappa}(Q) \times \mathcal{C}^{-\kappa}(Q) \to \mathbb{R}$ be defined by*

$$\Phi(Y, \zeta) := \inf_{v \in C_c^\infty : \|v\|_{L^2}=1} \int_Q e^{-2Y} \left(|\nabla v|^2 + \zeta |v|^2 \right) \mathrm{d}x$$

Then the map Φ is continuous and the lowest eigenvalue of Anderson Hamiltonian $-\Delta+\xi$ on Q_L with Dirichlet b.c. is given by $\lambda_1(Q_L, \xi) = \Phi(G * \xi, Z - F * \xi)$. In particular, the limit holds: $\lim_{\varepsilon \to 0} \lambda_1(Q_L, \xi_\varepsilon) = \lambda_1(Q_L, \xi)$

Proof. Let (Y_1, ζ_1), (Y_2, ζ_2) be two couples in $\mathcal{C}^{1-\kappa}(Q) \times \mathcal{C}^{-\kappa}(Q)$. One has

$$\begin{aligned} &\left| \int_Q e^{-2Y_1} \left(|\nabla v|^2 + \zeta_1 |v|^2 \right) - \int_Q e^{-2Y_2} \left(|\nabla v|^2 + \zeta_2 |v|^2 \right) \right| \\ &\lesssim \|e^{-2Y_1} - e^{-2Y_2}\|_{\mathcal{C}^{1-\kappa}} \left(\|\nabla v\|_{L^2}^2 + \|\zeta_1\|_{\mathcal{C}^{-\kappa}} \|v\|_{\mathcal{H}^1}^2 \right) + \|e^{-2Y_2}\|_{\mathcal{C}^{1-\kappa}} \|\zeta_1 - \zeta_2\|_{\mathcal{C}^{-\kappa}} \|v\|_{\mathcal{H}^1}^2 \end{aligned}$$

Notice that $\inf_{v \in C_c^\infty : \|v\|_{L^2}=1} \int_Q \|\nabla v\|^2 = \lambda_1(Q)$ where $\lambda_1(Q)$ is the bottom of spectrum of the Laplacian $-\Delta$ with Dirichlet boundary condition on Q. We thus have

$$|\Phi_1(Y_1, \zeta_1) - \Phi_2(Y_2, \zeta_2)| \lesssim (1+\lambda_1(Q)) \left(\|e^{-2Y_1} - e^{-2Y_2}\|_{\mathcal{C}^{1-\kappa}} \|\zeta_1\|_{\mathcal{C}^{-\kappa}} + \|e^{-2Y_2}\|_{\mathcal{C}^{1-\kappa}} \|\zeta_1 - \zeta_2\|_{\mathcal{C}^{-\kappa}} \right)$$

which proves the continuity. □

Lemma IV.25. *On a bounded box Q, the sequence of random variable $|\nabla Y_\varepsilon|^2 - c_\varepsilon$ converges in $\mathcal{C}^{-\kappa}(Q)$ to a random variable denoted by $: |\nabla Y|^2 :$. For every test function η, $\langle : |\nabla Y|^2 :, \eta \rangle$ is the second-order homogeneous Wiener chaos of ξ associated to the $L^2(\mathrm{d}z\, \mathrm{d}\bar{z})$-function given by*

$$(z, \bar{z}) \mapsto \int_Q \eta(x) \nabla G(x - z) \cdot \nabla G(x - \bar{z})\, \mathrm{d}x$$

On the abstract Wiener space $(\mathcal{C}^{-1-\kappa}, L^2, \mu)$ on which lives the white noise ξ, we introduce the functions: $\Psi_1(\xi) = G * \xi$, $\Psi_2(\xi) = -F * \xi$ and $\Psi_3(\xi) = - : |\nabla G * \xi|^2 :$ for $\xi \in \mathcal{C}^{-1-\kappa}$.

Lemma IV.26. *The triplet $\Psi(\xi) := (\Psi_1(\xi), \Psi_2(\xi), \Psi_3(\xi))$ induces a probability measure on the separable Banach space $\mathcal{C}^{1-\kappa} \times \mathcal{C}^\infty \times \mathcal{C}^{-\kappa}$. As $\beta \to 0$, the sequence $(\Psi(\beta^{1/2}\xi))_{\beta > 0}$ follows a large deviation principle with rate β and a good rate function*

$$I_\Psi(f_1, f_2, f_3) := \inf \left\{ \frac{1}{2} \|\varphi\|_{L^2}^2 : \varphi \in L^2, G * \varphi = f_1, -F * \varphi = f_2, -|\nabla G * \varphi|^2 = f_3 \right\}$$

for $f_1 \in \mathcal{C}^{1-\kappa}, f_2 \in \mathcal{C}^\infty$ and $f_3 \in \mathcal{C}^{-\kappa}$, with the convention $\inf \emptyset = +\infty$.

Proof. Let E be the separable Banach space $\mathcal{C}^{1-\kappa} \times \mathcal{C}^\infty \times \mathcal{C}^{-\kappa}$. We observe that Ψ is a random variable of class $L^2(\mu; E)$ and that Ψ_1, Ψ_2 are first-order Wiener chaos while Ψ_3 is a second-order Wiener chaos. By Cameron-Martin theorem, one can define the homogenous part of the map Ψ by

$$\Psi_{\text{hom}}(\varphi) := \int \Psi(\xi + \varphi) \mu(\mathrm{d}\xi) = (G * \varphi, -F * \varphi, -|\nabla G * \varphi|^2), \quad \varphi \in L^2$$

To prove the assertion, we will show that the sequence $(\Psi(\beta^{1/2}\xi))_{\beta>0}$ follows large deviation principle with rate β and rate function $I_\Psi(f_1, f_2, f_3) := \inf\{\frac{1}{2}\|\varphi\|^2_{L^2} : \varphi \in L^2, \Psi_{\text{hom}}(\varphi) = (f_1, f_2, f_3)\}$.

The proof of the claim follows from the same reasoning as in [HW15]: Let $(e_i)_{i\geq 1}$ be an orthonormal basis of L^2 consisting of elements in $(\mathcal{C}^{-1-\kappa})^*$ and let $\mathcal{F}_N$ be a σ-algebra on $\mathcal{C}^{-1-\kappa}$ generated by the random variables $\langle \xi, e_i\rangle$, $i = 1, \ldots, N$. Define $\Psi^{(N)}(\xi)$ to be the conditional expection of $\Psi(\xi)$ with respect to the σ-algebra $\mathcal{F}_N$.

We now verify that $\mathbb{E}[\Psi_j(\xi)|\mathcal{F}_N], j = 1, 2, 3$ are continuous functions of ξ. Indeed, we have $\mathbb{E}[\Psi_1(\xi)|\mathcal{F}_N] = G * \xi_N$ and $\mathbb{E}[\Psi_1(\xi)|\mathcal{F}_N] = -F * \xi_N$, where $\xi_N = \sum_{i=1}^N \langle \xi, e_i\rangle e_i$. Also, we have

$$|\nabla G_\varepsilon * \xi|^2 = |\nabla G_\varepsilon * (\xi - \xi_N)|^2 + 2\nabla G_\varepsilon * (\xi - \xi_N) \cdot \nabla G_\varepsilon * \xi_N + |\nabla G_\varepsilon * \xi_N|^2$$

By the fact that $\xi - \xi_N$ is independent of $\mathcal{F}_N$, upon taking conditional expectation we get

$$\mathbb{E}[|\nabla G_\varepsilon * \xi|^2|\mathcal{F}_N] = |\nabla G_\varepsilon * \xi_N|^2 + c_{\varepsilon,N},$$

where

$$c_{\varepsilon,N} = \mathbb{E}[|\nabla G_\varepsilon * (\xi - \xi_N)|^2] = \sum_{i,j>N} \nabla G_\varepsilon * e_i \cdot \nabla G_\varepsilon * e_j\, \mathbb{E}[\langle \xi, e_i\rangle \langle \xi, e_j\rangle] = \sum_{i>N} |\nabla G_\varepsilon * e_i|^2.$$

Recall that $|\nabla G_\varepsilon * \xi|^2 - c_\varepsilon$ converge in $L^2(\mu; \mathcal{C}^{-\kappa})$ to $- : |\nabla G * \xi|^2 :$ as $\varepsilon \downarrow 0$ with the constant $c_\varepsilon = \mathbb{E}[|\nabla G_\varepsilon * \xi|^2] = \sum_{i=1}^\infty |\nabla G_\varepsilon * e_i|^2$. Notice that $c_\varepsilon - |\nabla G_\varepsilon * \xi|^2$ converge in $L^2(\mu; \mathcal{C}^{-\kappa})$ to $- : |\nabla G * \xi|^2 :$ as $\varepsilon \downarrow 0$. We deduce therefore

$$\mathbb{E}[\Psi_3(\xi)|\mathcal{F}_N] = \sum_{i=1}^N |\nabla G * e_i|^2 - |\nabla G * \xi_N|^2$$

which is indeed a continuous function of ξ

Recall that $(\beta^{1/2}\xi)_{\beta>0}$ follows a large deviation principle with rate β and rate function $I_\xi(\varphi) = \frac{1}{2}\|\varphi\|^2_{L^2}$ if $\varphi \in L^2$ and $= \infty$ otherwise. By contraction principle, we infer $\Psi^{(N)}(\beta^{1/2}\xi)$ satisfies a large deviation principle with rate β and rate function $I^{(N)}_\Psi(f_1, f_2, f_3) = \inf\{\frac{1}{2}\|\varphi\|^2_{L^2} : \varphi \in L^2, \Psi^{(N)}(\varphi) = (f_1, f_2, f_3)\}$ □

Proposition IV.27 (Large deviation)**.** *Fix $L > 0$.*

1. *The sequence of random variables $(\lambda(Q_L, \beta^2\mathbf{A}, \beta^{2-d/2}\xi) + \delta_\beta)_{\beta>0}$ is exponentially equivalent to $(\lambda(Q_L, \beta^{2-d/2}\xi))_{\beta>0}$.*

2. *$(\lambda(Q_L, \beta^2\mathbf{A}, \beta^{2-d/2}\xi) + \delta_\beta)_{\beta>0}$ satisfies a large deviation principle with rate β^{4-d} with the good rate function*

$$I_L(x) = \inf\left\{\frac{1}{2}\|\varphi\|^2_{L^2} : \varphi \in L^2(Q_L) \text{ such that the lowest e.v. of } -\Delta + \varphi = x\right\}$$

Proof. 1. By Propition IV.22 $\delta_\beta \to 0$ as $\beta \to 0$ deterministically, the constant δ_β can thus be dropped by exponential equivalence.
Now consider the case $V_\varepsilon = \beta^{2-d/2}\xi_\varepsilon + c_\varepsilon(\beta^{2-d/2})$. Denote $G_Q^{\beta^2\mathbf{A},V_\varepsilon}(u) := \int_Q |(i\nabla + \beta^2\mathbf{A})u|^2 + V_\varepsilon|u|^2$ and $G_Q^{V_\varepsilon}(u) := \int_Q |\nabla u|^2 + V_\varepsilon|u|^2$. Denote also $m_\beta := \|\beta^2\mathbf{A}\|_{L^\infty(Q)} = cB|Q|^{\frac{1}{2}}\beta^2$ for some constant c. We see, for all $u \in C_c^\infty$ such that $\|u\|_{L^2(Q)} = 1$,

$$|G_Q^{\beta^2\mathbf{A},V_\varepsilon}(u) - G^{V_\varepsilon}(u)| = \int_Q (2i\beta^2\mathbf{A}\cdot\nabla u)\bar{u} + |\beta^2\mathbf{A}|^2|u|^2 \le m_\beta\|\nabla u\|^2_{L^2(Q)} + m_\beta + m_\beta^2.$$

It follows that

$$G_Q^{\mathbf{A},V_\varepsilon}(u) \begin{cases} \le (1+m_\beta)\int_Q |\nabla u|^2 + \int_Q V_\varepsilon|u|^2 + m_\beta + m_\beta^2 \\ \ge (1-m_\beta)\int_Q |\nabla u|^2 + \int_Q V_\varepsilon|u|^2 + m_\beta + m_\beta^2 \end{cases}$$

from which one has $|\lambda(Q,\beta^2\mathbf{A},V_\varepsilon) - \lambda(Q,V_\varepsilon)| \le m_\beta\lambda_1(Q) + m_\beta + m_\beta^2$. Passing $\varepsilon \to 0$ and then $\beta \to 0$, one deduces that $|\lambda(Q,\beta^2\mathbf{A},\xi) - \lambda(Q,\xi)| \to 0$ deterministically as $\beta \to 0$. The exponential equivalence then follows.

2. This is a consequence of point 1, Lemma IV.24,IV.26 and the contraction principle: indeed, the eigenvalue $\lambda(Q_L,\beta^{1/2}\xi)$ can be writen as $\Phi \circ g \circ \Psi(\beta^{1/2}\xi)$ where Φ is defined in Lemma IV.24 and the function $g : \mathcal{C}^{1-\kappa} \times \mathcal{C}^\infty \times \mathcal{C}^{-\kappa} \to \mathcal{C}^{1-\kappa} \times \mathcal{C}^{-\kappa}$ is defined by $g(x_1,x_2,x_3) = (x_1, x_2 + x_3)$. As $\Phi \circ g$ is continuous, contraction principle implies that $\lambda(Q_L,\beta^{1/2}\xi)$ follows a large deviation principle with rate β and rate function I_L given in the statement. By the exponential equivalence, $(\lambda(Q_L,\beta^2\mathbf{A},\beta^{2-d/2}\xi) + \delta_\beta)_{\beta>0}$ follows the same LDP with $(\lambda(Q_L,\beta^{2-d/2}\xi))_{\beta>0}$, i.e. with rate β^{4-d} and rate function I_L.

□

IV.4.2 Tail bound and asymptotics

Theorem IV.28. *Fix $L \ge 1$, $\eta \in (0,1)$ and $n \ge 1$. There exist $r > 0$ and $x_0 \ge (L/r)^2$ such that the following inequalities hold: for all $x \ge x_0$ we have*

$$r^{-d}x^{d/2}e^{d\log L-(1+\eta)\rho x^{2-d/2}} \le \mathbb{P}(-\lambda_n(Q_L,\mathbf{A},\xi) \ge x) \le r^{-d}x^{d/2}e^{d\log L-\rho(1-\eta)x^{2-d/2}} \quad \text{(IV.20)}$$

with $\rho = \inf_{r>0}\inf\{I_r(x) : x \in (-\infty,-1)\}$.

Moreover, the constant ρ satisfies $\rho = \frac{d}{C}$ with C being the optimal constant of Ladyzhenskaya inequality.

Proof. We follow the same line of proof as in [HL22]. First consider the case $n = 1$. Fix $r > 0$ large enough (to be specified later) and denote $\beta = 1/\sqrt{x}$. For $x \ge (L/r)^2$, we have $0 < r \le L/\beta$, and by Proposition IV.22, IV.23 as well as the independence of ξ on disjoint boxes, one deduces

$$\begin{aligned}
\mathbb{P}(-\lambda_1(Q_L,\mathbf{A},\xi)\geq x) &= \mathbb{P}(\beta^2\lambda_1(Q_L,\mathbf{A},\xi)\leq -1)\\
&= \mathbb{P}(\lambda_1(Q_{L/\beta},\beta^2\mathbf{A},\beta^{2-d/2}\xi)+\delta_\beta\leq -1)\\
&\begin{cases}\leq \displaystyle\sum_{k\in\mathbb{Z}^2:|k|_\infty<\frac{L}{2\beta r}+\frac{3}{2}} \mathbb{P}\left(\lambda(kr+Q_{\frac{3r}{2}},\beta^2\mathbf{A},\beta^{2-d/2}\xi)-\frac{K}{r^2}+\delta_\beta\leq -1\right)\\
\geq 1-\displaystyle\prod_{k\in\mathbb{Z}^2:|k|_\infty<\frac{L}{2\beta r}} \left[1-\mathbb{P}\left(\lambda_1(kr+Q_r,\beta^2\mathbf{A},\beta^{2-d/2}\xi)+\delta_\beta\leq -1\right)\right]\end{cases}
\end{aligned}$$

Recall from the point 1. of Proposition IV.22 that the eigenvalues are invariant in law under translation of domain, and note that $|\{k\in\mathbb{Z}^2:|k|_\infty<\frac{L}{2\beta r}\}|=c_1(\frac{L}{\beta r})^d$ and $|\{k\in\mathbb{Z}^2:|k|_\infty<\frac{L}{2\beta r}+\frac{3}{2}\}|=c_2(\frac{L}{\beta r})^d$ for some constants $c_1\leq c_2$. Denote $p_r(\beta)=\mathbb{P}(\lambda_1(Q_r,\beta^2\mathbf{A},\beta^{2-d/2}\xi)+\delta_\beta\leq -1)$ and $q_r(\beta)=\mathbb{P}(\lambda_1(Q_{\frac{3r}{2}},\beta^2\mathbf{A},\beta^{2-d/2}\xi)-\frac{K}{r^2}+\delta_\beta\leq -1)$. We thus have

$$1-(1-p_r(\beta))^{c_1\left(\frac{L}{\beta r}\right)^d}\leq \mathbb{P}(-\lambda_1(Q_L,\mathbf{A},\xi)\geq x)\leq c_2\left(\frac{L}{\beta r}\right)^d q_r(\beta)$$

By the large deviation principle satisfied by $\lambda_1(Q_r,\beta^2\mathbf{A},\beta^{2-d/2}\xi)+\delta_\beta$ (and the fact that the sequence converges to the smallest eigenvalue $\lambda_1(Q_r)$ of $-\Delta$ on Q_r, which is positive), $p_r(\beta)$ is the probability of a large deviation event: $p_r(\beta)$ decreases exponentially as $\beta\to 0$. It follows that

$$1-(1-p_r(\beta))^{c_1\left(\frac{L}{\beta r}\right)^d}=c_1\left(\frac{L}{\beta r}\right)^d p_r(\beta)(1+o(1)),\quad \beta\to 0.$$

In particular, one has

$$\log c_1+\log p_r(\beta)+o(1)\leq \log\left[\left(\frac{\beta r}{L}\right)^d\mathbb{P}(-\lambda_1(Q_L,\mathbf{A},\xi)\geq x)\right]\leq \log c_2+\log q_r(\beta)$$

Using the large deviation principle for the eigenvalues (Proposition IV.27), we obtain

$$\limsup_{\beta\to 0}\beta^{4-d}\log\left[\left(\frac{\beta r}{L}\right)^d\mathbb{P}(-\lambda_1(Q_L,\mathbf{A},\xi)\geq x)\right]\leq -\inf\left\{I_{\frac{3r}{2}}(x):x\in(-\infty,-1+\tfrac{K}{r^2}]\right\} \tag{IV.21}$$

$$\liminf_{\beta\to 0}\beta^{4-d}\log\left[\left(\frac{\beta r}{L}\right)^d\mathbb{P}(-\lambda_1(Q_L,\mathbf{A},\xi)\geq x)\right]\geq -\inf\left\{I_r(x):x\in(-\infty,-1)\right\} \tag{IV.22}$$

Let $\rho=\inf_{r>0}\inf\{I_r(x):x\in(-\infty,-1)\}$. Clearly we have $\rho\leq\inf\{I_{\frac{3r}{2}}(x):x\in(-\infty,-1)\}\leq\inf\{I_{\frac{3r}{2}}(x):x\in(-\infty,-1+\frac{K}{r^2}]\}$ for any $r>0$. Moreover, given $\eta>0$, we can find a r large enough so that $\rho(1+\eta/2) \geq \inf\{I_r(x) : x \in (-\infty, -1)\}$. In particular, for such $r > 0$, one has

$$\limsup_{\beta \to 0} \beta^{4-d} \log \left[\left(\frac{\beta r}{L} \right)^d \mathbb{P}(-\lambda_1(Q_L, \mathbf{A}, \xi) \geq x) \right] < -\rho(1-\eta) \tag{IV.23}$$

$$\liminf_{\beta \to 0} \beta^{4-d} \log \left[\left(\frac{\beta r}{L} \right)^d \mathbb{P}(-\lambda_1(Q_L, \mathbf{A}, \xi) \geq x) \right] > -\rho(1+\eta) \tag{IV.24}$$

from which the assertion for $n = 1$ follows.

For general $n \geq 1$, notice that

$$\lambda_1(Q_L, \mathbf{A}, \xi) \leq \lambda_n(Q_L, \mathbf{A}, \xi) \leq \max_{i=1,\dots,n} \lambda_1(Q^{(i)}, \mathbf{A}, \xi)$$

where $Q^{(i)}, i = 1, \dots, n$ are n disjoint open boxes contained in Q_L. It then suffices to argue that both sides satisfy the same asymptotics the same tail bound. □

Appendix A

Strong resolvent convergence of Anderson Hamiltonian from finite volume to full space

In this section, we consider the continuous Anderson Hamiltonian $\mathcal{H}$ (i.e., $\mathbf{A} = 0$ and $V = \xi$ in (I.1)) for $d \in \{2,3\}$. Let $\mathcal{H} = -\Delta + \xi$ the Anderson Hamiltonian constructed in Chapter III as a self-adjoint unbounded operator acting on $L^2(\mathbb{R}^d)$. Denote by $\mathcal{H}_L$ the Anderson Hamiltonian on the box $Q_L = (-L/2, L/2)^d$ with zero Dirichlet boundary condition (whose construction can be carried out by exactly the same argument in Chapter III, just by replacing $\mathbb{R}^d$ with Q_L and removing the weights; one however has to be careful about the Dirichlet boundary condition: this can be taken into account by modifying the Green function for the heat kernel, see for example [Lab19]).

The aim is to prove the following statement.

Theorem A.1. *$\mathcal{H}_L$ converges to $\mathcal{H}$ in strong resolvent sense as $L \to \infty$, in probability.*

From Lemma III.14 and the equivalence between strong and weak resolvent convergence ([Tes14, Lemma 6.37]), we have the following fact.

Proposition A.2. *Let $S_n, n = 1, 2, \ldots,$ and S be self-adjoint operators. Let E_n and E denote the projection-valued measures of S_n, S, respectively. Then, S_n converges to S in strong resolvent sense as $n \to \infty$ if and only if there exist some $T > 0$ such that*

$$\int_{\mathbb{R}} e^{-\lambda t} \langle E_n(\mathrm{d}\lambda) f, f \rangle \xrightarrow[n\to\infty]{} \int_{\mathbb{R}} e^{-\lambda t} \langle E(\mathrm{d}\lambda) f, f \rangle, \quad \forall t \in [0,T],$$

for all $f \in L^2(\mathbb{R}^d)$ such that the above integrals are finite.

Proof of Theorem A.1. Recall that $C_c^\infty(\mathbb{R}^d)$ is dense in the domain of $e^{-t\mathcal{H}}$ and that of $e^{-t\mathcal{H}_L}$. By virtue of Proposition A.2, it suffices to show that for some $T > 0$,

$$\left\langle e^{-t\mathcal{H}_L} f, f \right\rangle_{L^2} \xrightarrow[L\to\infty]{} \left\langle e^{-t\mathcal{H}} f, f \right\rangle_{L^2},$$

for all $f \in C_c^\infty(\mathbb{R}^d)$ and all $t \in [0,T]$, in probability.

Let us introduce the operators $\mathcal{H}_\varepsilon$ and $\mathcal{H}_{L,\varepsilon}$ associated to the regularized noise $\xi_\varepsilon = \rho_\varepsilon * \xi$, which are defined on $L^2(\mathbb{R}^d)$, $L^2(Q_L)$, respectively. By classical operator theory (see [Kat72] for example), it is clear that both $\mathcal{H}_\varepsilon$ and $\mathcal{H}_{L,\varepsilon}$ essentially self-adjoint (on $C_c^\infty(\mathbb{R}^d)$ and $C_c^\infty(Q_L)$).

Now fix $f \in C_c^\infty$ and suppose L is large enough so that f is supported in Q_L. Let us write $u^f(t,x) = e^{-t\mathcal{H}}f(x)$, $u_L^f(t,x) = e^{-t\mathcal{H}_L}f(x)$, which are solutions to the parabolic equations with initial data f, associated to the operators $\mathcal{H}$, $\mathcal{H}_L$, respectively. Recall from Chapter III that they are the limits (in probability) of $u_\varepsilon^f(t,x) = e^{-t\mathcal{H}_\varepsilon}f(x)$, $u_{L,\varepsilon}^f(t,x) = e^{-t\mathcal{H}_{L,\varepsilon}}f(x)$ in the following sense

$$\left\langle u_\varepsilon^f, f\right\rangle_{L^2} \to \left\langle u^f, f\right\rangle, \quad \left\langle u_{L,\varepsilon}^f, f\right\rangle_{L^2} \to \left\langle u_L^f, f\right\rangle, \qquad \text{as } \varepsilon \to 0.$$

Moreover, the above convergence is uniform for all L, since the norm of the stochasitc objects (i.e., the $Q_\varepsilon(\xi)$ in dimension 2 and the models in dimension 3) are uniformly bounded for all L.

Fix $\delta > 0$, there exists ε_0 small enough such that $|\left\langle u_{\varepsilon_0}^f, f\right\rangle_{L^2} - \left\langle u^f, f\right\rangle| + |\left\langle u_{L,\varepsilon_0}^f, f\right\rangle_{L^2} - \left\langle u_L^f, f\right\rangle_{L^2}| \leq \delta$ uniformly for all L. By triangular inequality, one has

$$\left|\left\langle e^{-t\mathcal{H}_L}f, f\right\rangle_{L^2} - \left\langle e^{-t\mathcal{H}}f, f\right\rangle_{L^2}\right| \leq \left|\left\langle e^{-t\mathcal{H}_{L,\varepsilon_0}}f, f\right\rangle_{L^2} - \left\langle e^{-t\mathcal{H}_{\varepsilon_0}}f, f\right\rangle_{L^2}\right| + \delta.$$

Note that for every fixed $f \in C_c^\infty(\mathbb{R}^d)$ and all L large enough, $\mathcal{H}_{L,\varepsilon_0}f$ agrees with $\mathcal{H}_{\varepsilon_0}f$. Since $C_c^\infty(\mathbb{R}^d)$ is a core for the operator $\mathcal{H}_{\varepsilon_0}$, the criterion of strong resolvent convergence due to Weidmann [Wei97] implies that $\mathcal{H}_{L,\varepsilon_0}$ converges to $\mathcal{H}_{\varepsilon_0}$ in strong resolvent sense when $L \to \infty$. This in turn implies (by the weak convergence of the spectral measure associated to f)

$$\left|\left\langle e^{-t\mathcal{H}_{L,\varepsilon_0}}f, f\right\rangle_{L^2} - \left\langle e^{-t\mathcal{H}_{\varepsilon_0}}f, f\right\rangle_{L^2}\right| \to 0$$

as $L \to \infty$. Hence,

$$\limsup_{L\to\infty} \left|\left\langle e^{-t\mathcal{H}_L}f, f\right\rangle_{L^2} - \left\langle e^{-t\mathcal{H}}f, f\right\rangle_{L^2}\right| \leq \delta.$$

Since δ is arbitrary, this thus concludes the proof. $\square$

Annexe B

Résumé des chapitres en français

B.1 CHAPITRE I : INTRODUCTION

B.1.1 LE CONTEXTE

Cette thèse tourne autour de l'opérateur de Schrödinger aléatoire, c'est-à-dire un opérateur différentiel prenant la forme

$$\mathcal{H} = (i\nabla + \mathbf{A})^2 + V, \tag{B.1}$$

où $\mathbf{A}$ est une fonction vectorielle et V est un potentiel aléatoire. L'intérêt pour de tels opérateurs remonte aux axiomes de la mécanique quantique, où les grandeurs physiques d'un système donné, telles que l'énergie, sont postulées être représentées par des opérateurs différentiels auto-adjoints, et ce que l'on peut réellement mesurer de ces grandeurs lors des expériences correspond aux valeurs spectrales des opérateurs associés. La procédure consistant à associer un opérateur différentiel à une quantité physique est appelée *quantification*. En effet, l'opérateur de la forme (B.1) est obtenu en quantifiant l'Hamiltonien (i.e., l'énergie totale) d'une particule soumise à un champ magnétique $\mathbf{B} = \operatorname{curl} \mathbf{A}$ et un potentiel électrique V. Inutile de dire qu'il est d'un grand intérêt pour les physiciens de comprendre les propriétés spectrales de (B.1).

Définir correctement (B.1) en tant qu'opérateur auto-adjoint n'est pas une tâche triviale. En termes mathématiques, cela consiste à choisir un espace de Hilbert $\mathscr{H}$ (généralement L^2) et un sous-espace approprié $\mathscr{D} \subset \mathscr{H}$ de manière à ce que l'application linéaire $\mathcal{H} : \mathscr{D} \to \mathscr{H}$ soit auto-adjointe. Ce type d'opérateur est appelé *non-borné* en raison du fait que son expression différentielle (B.1) n'est censée avoir un sens que sur un sous-espace propre $\mathscr{D}$ de $\mathscr{H}$, contrairement au cas des opérateurs bornés. La nature de l'opérateur $\mathcal{H}$ varie grandement avec le $\mathscr{D}$ donné, au sens où même si l'expression (B.1) pouvait être définie sur deux sous-espaces distincts $\mathscr{D}_1$ et $\mathscr{D}_2$, les propriétés spectrales de $\mathscr{H}$ sur chaque domaine pourraient être complètement différentes. Cette question de domaine est plus qu'une considération mathématique : la physique impose généralement des contraintes sur le choix de $\mathscr{D}$.

Étant donné un opérateur auto-adjoint $\mathcal{H}$, le théorème spectral (Théorème I.12) et le calcul fonctionnel (Théorème I.11) permettent de résoudre l'équation de Schrödinger

$$i\partial_t u = \mathcal{H}u, \quad u(0,\cdot) = \psi \in \mathscr{H},$$

dont la solution $u = e^{-it\mathcal{H}}\psi$ décrit l'évolution temporelle de la fonction d'onde de la particule (avec la condition initiale ψ) dans le système associé à l'Hamiltonien $\mathcal{H}$. Le comportement de u est en fait déterminé par la nature spectrale de $\mathcal{H}$. Plus précisément, le spectre d'un opérateur auto-adjoint $\mathcal{H}$ est un ensemble fermé $\sigma(\mathcal{H}) \subset \mathbb{R}$ qui peut être décomposé en

$$\sigma(\mathcal{H}) = \overline{\sigma_{\mathrm{pp}}(\mathcal{H})} \cup \sigma_{\mathrm{c}}(H).$$

Ici, $\sigma_{\mathrm{pp}}(\mathcal{H})$ désigne le *spectre purement ponctuel*, c'est-à-dire l'ensemble des valeurs propres de $\mathcal{H}$, tandis que $\sigma_{\mathrm{c}}(\mathcal{H})$ représente le *spectre continu*. Chacun de ces spectres correspond à un sous-espace de $\mathscr{H}$ satisfaisant

$$\mathscr{H} = \mathscr{H}_{\mathrm{pp}} \oplus \mathscr{H}_{\mathrm{c}}.$$

Leurs définitions précises ne seront pas données ici, mais on peut garder à l'esprit l'image suivante donnée par le célèbre théorème RAGE : la solution u à l'équations de Schrödinger avec une fonction d'onde initiale $\psi \in \mathscr{H}_{\mathrm{pp}}$ restera concentrée dans une région bornée pendant l'évolution temporelle. En revanche, la masse de u correspondant à une fonction d'onde initiale $\psi \in \mathscr{H}_{\mathrm{c}}$ se dispersera dans tout l'espace. En termes de la physique, les valeurs spectrales dans σ_{pp} sont appelées les *états liés* (bound states) tandis que celles dans σ_{c} sont appelées les *états de diffusion* (scattering states) ou *états étendus* (extended states)[1]. En effet, la compréhension de la nature spectrale d'un opérateur de Schrödinger donné est d'une importance primordiale pour décrire la dynamique d'un système quantique.

L'introduction de l'aléa dans V remonte à l'étude des systèmes désordonnés, dont l'un des exemples les plus marquants est le modèle d'Anderson [And58] qui vise à expliquer la perte de conductivité lorsque des impuretés sont présentes dans un conducteur métallique. Anderson a considéré l'Hamiltonien

$$-\Delta_{\mathrm{d}} + \eta \tag{B.2}$$

défini sur l'espace de Hilbert $\ell^2(\mathbb{Z}^d)$, où Δ_{d} désigne le Laplacien discret sur le réseau $\mathbb{Z}^d$ et $\eta = (\eta(x))_{x \in \mathbb{Z}^d}$ est une collection de variables aléatoires i.i.d. (indépendantes et identiquement distribuées). Sous certaines conditions sur la loi de η, on prédit que le modèle d'Anderson possède les propriétés spectrales suivantes, connues sous le nom de *localisation d'Anderson* :

- En dimensions $d = 1, 2$, (B.2) a un spectre purement ponctuel $\sigma = \overline{\sigma_{\mathrm{pp}}}$ avec des fonctions propres exponentiellement décroissantes.
- En dimension $d \geq 3$, la localisation se produit pour des énergies suffisamment basses, c'est-à-dire qu'il existe un réel E suffisamment bas tel que $\sigma \cap (-\infty, E]$ est purement ponctuel avec des fonctions propres exponentiellement décroissantes.

[1] Dans certaines littératures, les états de diffusion ne font référence qu'au *spectre absolument continu* qui n'est qu'un sous-ensemble de σ_{c} ; la signification physique de la partie restante, le *spectre singulièrement continu*, est subtile et dépasse le cadre de la thèse.

La prédiction ci-dessus donne l'image suivante : En dimensions $d = 1, 2$, une fois que le potentiel aléatoire est activé (quelle que soit son intensité), la fonction d'onde reste localisée et donc la conductivité disparaît ; pour les dimensions supérieures $d \geq 3$, la conductivité disparaît à condition que l'énergie soit suffisamment basse par rapport à l'intensité du désordre. La localisation d'Anderson a suscité un grand intérêt dans la communauté mathématique et, dans le cas discret décrit ci-dessus, des preuves mathématiquement rigoureuses ont été obtenues dans de divers contextes. Pour un survol des résultats du cas discret, voir les notes de cours de Kirsch [Kir07] et les références qui y sont citées.

Des efforts ont également été déployés pour généraliser le cadre discret au cadre continu. Sur l'espace de Hilbert $\mathscr{H} = L^2(\mathbb{R}^d)$, nous considérons l'opérateur (B.1) avec un potentiel aléatoire V défini sur $\mathbb{R}^d$. Un choix naturel pour le potentiel dans ce cas est le champ gaussien stationnaire caractérisé par la fonction de covariance $C(x) = \mathbb{E}[V(0)V(x)]$, $x \in \mathbb{R}^d$. En raison de la non-bornitude de V, il est d'abord nécessaire de démontrer que cet opérateur est auto-adjoint sur un certain domaine : ceci est prouvé par [KM83].

L'objectif principal de cette thèse est d'étendre le cadre continu pour inclure des potentiels aléatoires à valeurs dans l'espace des distributions, et, à long terme, de vérifier si la prédiction d'Anderson décrite précédemment persiste dans ce cas. Un exemple majeur de cette classe est l'Hamiltonien d'Anderson continu

$$\mathcal{H} = -\Delta + \xi \tag{B.3}$$

qui émerge comme la limite d'échelle formelle de l'opérateur discret (B.1). Ici, Δ est le Laplacien sur $\mathbb{R}^d$ et ξ est le bruit blanc gaussien spatial. Plus précisément, le bruit blanc gaussien ξ est la limite en loi du champ aléatoire mis à l'échelle $\xi_h(x) = h^{d/2}\eta(x/h)$, $x \in h\mathbb{Z}^d$, lorsque $h \to 0$, où $(\eta(k))_{k\in\mathbb{Z}^d}$ est une famille de variables aléatoires i.i.d. définie dans (B.2). Le bruit blanc peut également être vu comme un champ gaussien stationnaire avec une fonction de covariance donnée par la distribution de Dirac, $C(x) = \delta(x)$. Plus précisément, ξ est la famille gaussienne $(\xi(\varphi))$ indexée par les fonctions tests liesses à supports compacts $\varphi \in C_c^\infty(\mathbb{R}^d)$ avec la fonction de covariance

$$\mathbb{E}[\xi(\varphi_1)\xi_1(\varphi_2)] = \langle \varphi_1, \varphi_2 \rangle_{L^2} .$$

En effet, le membre droit de cette identité est la définition rigoureuse de $\iint \varphi(x)\delta(x - y)\varphi(y)\, \mathrm{d}x\, \mathrm{d}y$, qui justifie l'expression formelle $C = \delta$. En fait, presque sûrement,le bruit blanc gaussien ξ est seulement une distribution et a une régularité Hölder négative de $-d/2 - \varepsilon$, d étant la dimension spatiale (voir Définition I.24 et Remarque I.25). En raison de cette singularité, on ne peut définir le produit $\xi \cdot f$ que si la fonction f est suffisamment régulière (en fait, plus régulière que $(d/2 + \varepsilon)$-Hölder). Pour cette raison, la théorie classique des opérateurs échoue à identifier un domaine sur lequel (B.3) est bien défini, et par conséquent, un sens mathématiquement rigoureux de (B.3) en tant qu'opérateur auto-adjoint a été inaccessible jusqu'à récemment. Nous désignerons ces opérateurs classiquement mal posés par le terme *opérateurs de Schrödinger aléatoires singuliers* (Définition I.26).

Une manière de donner un sens mathématiquement rigoureux à ces opérateurs est d'utiliser les nouvelles théories des EDPS singulières. Deux des avancées les plus importantes dans ce domaine sont la théorie des structures de régularité [Hai14] dûe à Hairer et la théorie des distributions paracontrôllées développée par Gubinelli, Imkeller et Perkowski [GIP15]. Dans la section I.3.2, nous expliquerons la méthodologie des EDPS pour construire des opérateurs singuliers, et les lecteurs trouveront dans la section I.3.4 un survol des avancées récentes dans le domaine. En particulier, la théorie des structures de régularité joue un rôle essentiel dans cette thèse. Pour cette raison, nous consacrons la section I.4 à son introduction : l'objectif est d'illustrer son cadre (Figure I.1) et son application à la construction d'opérateurs de Schrödinger singuliers aléatoires.

Une autre considération dans notre cadre est la présence du potentiel vectoriel magnétique $\mathbf{A}$. La motivation d'inclure ce terme magnétique provient de la différence spectrale remarquable entre l'opérateur $(i\nabla+\mathbf{A})^2$ et le Laplacien $-\Delta$ sur l'espace complet $\mathbb{R}^d$. Ici, l'opérateur $(i\nabla+\mathbf{A})^2$ est appelé l'*Hamiltonien de Landau* ou le *Laplacien magnétique*, car il résulte de la quantification de l'énergie cinétique d'une particule soumise à un champ magnétique. Par exemple, si l'on considère une particule placée sur le plan xy sur lequel un champ magnétique de magnitude $B > 0$ est appliqué dans la direction z, alors le potentiel magnétique $\mathbf{A}$ peut être défini comme[2]

$$\mathbf{A}(x_1, x_2) = \frac{B}{2}(-x_2, x_1), \quad (x_1, x_2) \in \mathbb{R}^2. \tag{B.4}$$

Avec ce choix de $\mathbf{A}$, le spectre de l'Hamiltonien de Landau $(i\nabla+\mathbf{A})^2$ en tant qu'opérateur non-borné auto-adjoint sur $L^2(\mathbb{R}^2)$ est donné par

$$\{(2n+1)B : n = 0, 1, 2, \dots\},$$

où chaque $(2n+1)B$ est une valeur propre de multiplicité infinie, appelée *niveau de Landau*. Comme on peut le voir, ce spectre est très différent de celui du Laplacien $-\Delta$, qui correspond à la demi-droite non négative $[0, +\infty)$. Par conséquent, une question naturelle se pose : comment le bruit blanc perturberait-il le spectre de l'Hamiltonien de Landau ? Notre objectif est de poursuivre cette direction de recherche.

Dans cette thèse, nous ne considérerons que les potentiels vectoriels magnétiques de la forme (B.4) pour la dimension $d = 2$, ou leur analogue pour $d = 3$,

$$\mathbf{A}(x_1, x_2, x_3) = \frac{B}{2}(-x_2, x_1, 0), \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \tag{B.5}$$

bien que nous puissions probablement envisager des potentiels beaucoup plus généraux dans $L^2_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^d)$.

[2]Notez que, pour un champ magnétique $\mathbf{B}$ donné, le choix d'un potentiel magnétique correspondant $\mathbf{A}$ n'est pas unique. En effet, si $\operatorname{curl} \mathbf{A} = \mathbf{B}$, alors pour toute fonction scalaire différentiable deux fois f, $\operatorname{curl}(\mathbf{A} + \nabla f) = \mathbf{B}$. Cependant, ces choix différents de $\mathbf{A}$ conduisent tous à des opérateurs unitairement équivalents, et donc à la même *physique*.

Enfin, notons qu'il est plus intéressant de considérer l'opérateur magnétique dans l'espace euclidien complet plutôt que dans un espace de volume borné tel que le tore $\mathbb{T}^d$ (qui serait beaucoup plus propice à l'analyse par les théories des EDPS singulières) : pour l'Hamiltonien de Landau non perturbé avec $\mathbf{A}$ donné par (B.4), les niveaux de Landau n'apparaissent que sur l'espace complet ; sur un espace de volume borné, $-\Delta$ et $(i\nabla+\mathbf{A})^2$ ont des formes quadratiques équivalentes et chacun admet un spectre purement discret avec des valeurs propres de multiplicité finie.

B.1.2 aperçu de la thèse

La présente thèse est consacrée à deux objectifs généraux :

1. Construire $\mathcal{H}$ représenté dans (B.1) sous différents contextes en tant que variable aléatoire *auto-adjointe à valeurs opérateurs.*
2. Comprendre le spectre (a priori aléatoire) de $\mathcal{H}$.

Rappelons le programme décrit dans la Figure I.1, qui peut être résumé par le diagramme suivant :

$$\begin{array}{ccccccc} \Omega & \xrightarrow{(a)} & \mathscr{M} & \xrightarrow{(b)} & \text{Solution à (I.11) ou (I.12)} & \xrightarrow{(c)} & \text{Operateur} \\ \xi & \mapsto & (\Pi,\Gamma) & \mapsto & u & \mapsto & \mathcal{H} \end{array}$$

Figure B.1 : Résumé de la Figure I.1.

Comme mentionné précédemment, les applications (b) et (c) sont continues : La continuité de (b) est une conséquence de la continuité de l'opérateur de reconstruction (section I.4.1) et de celle du point fixe par rapport aux modèles (section I.4.3) ; La continuité de (c) peut être prouvée par (une certaine généralisation de) les idées de la section I.3.2. D'autre part, l'application (a) est seulement mesurable. À partir de ces observations, l'objectif 1. peut être atteint si nous remplissons le programme $(a), (b), (c)$.

Pour aborder l'objectif 2., nous considérons un modèle admissible $Z = (\Pi, \Gamma) \in \mathscr{M}$ comme une variable aléatoire définie sur l'espace de probabilité canonique Ω du bruit blanc ξ. En étudiant la loi du modèle, on peut espérer retrouver certaines propriétés spectrales de l'opérateur $\mathcal{H}$ grâce à la continuité des applications (b) et (c).

Ce qui suit présente les résultats exposés dans cette thèse, obtenus en utilisant ce raisonnement général.

Chapitre II : Asymptotique des valeurs propres d'Anderson

Ce chapitre est basé sur le travail conjoint avec C. Labbé [HL22], publié dans *Stochastics and Partial Differential Equations : Analysis and Computations*, Volume 11, Numéro 3, septembre 2023.

Nous étudions l'Hamiltonien d'Anderson continu $\mathcal{H}_L$, c'est-à-dire, avec $\mathbf{A} = 0$ et $V = \xi$ dans (I.1), défini sur $L^2((-L/2, L/2)^d)$ pour $d = 1, 2, 3$. Sa construction est fournie par [Lab19], qui combine l'idée de *mild solution* (section I.3.2, Construction I) et de la structure de régularité décrite dans la section I.4. Cet argument affirme que $\mathcal{H}_L$ est une variable aléatoire à valeurs opérateurs auto-adjointe, et l'opérateur $\mathcal{H}_L$ a presque sûrement un résolvant compact. Par conséquent, il admet un spectre purement discret :

$$\lambda_{1,L} \leq \lambda_{2,L} \leq \lambda_{3,L} \leq \cdots .$$

Le principal résultat de ce travail concerne l'asymptotique presque sûre des valeurs propres $\lambda_{n,L}$ lorsque $L \to \infty$.

Theorem B.1 (Theorem II.1)**.** *Fixons $d \in 1, 2, 3$ et $n \geq 1$. Il est presque sûr que*

$$\lambda_{n,L} = -(C_d \log L)^{\frac{1}{2-d/2}}(1 + o(1)), \quad L \to \infty. \tag{B.6}$$

De plus, la constante C_d est explicite : elle dépend uniquement de la dimension d et est étroitement liée à la constante optimale de l'inégalité de Gagliardo-Nirenberg.

En dimension $d = 1$, la constante C_1 vaut $\frac{8}{3}$, ce qui conduit à l'asymptotique $\lambda_{n,L} \sim (\frac{8}{3} \log L)^{2/3}$, correspondant au résultat obtenu par McKean [McK94] mentionné dans la section I.3.4. En dimension $d = 2$, l'asymptotique se lit $\lambda_{n,L} \sim C_2 \log L$ avec

$$C_2 = \sup_{f \in H^1 \setminus \{0\}} \frac{\|f\|_{L^4}^4}{\|\nabla f\|_{L^2}^2 \|f\|_{L^2}^2},$$

ce qui correspond au résultat de Chouk et van Zuijlen [Cv21]. En dimension $d = 3$, notre résultat est nouveau.

Exposons les principaux arguments de la preuve. Nous avons d'abord besoin de quelques résultats de mise à l'échelle qui relient les valeurs propres $\lambda_{n,L}$ à l'opérateur associé au bruit $\beta\xi$, avec un préfacteur $\beta > 0$. Notons par $Z(\beta)$ le modèle renormalisé construit à partir de $\beta\xi$. Rappelons que $(Z(\beta))_{\beta>0}$ est une famille de variables aléatoires prenant des valeurs dans $\mathscr{M}$, l'espace des modèles admissibles. Lorsque $\beta \to 0$, Hairer et Weber [HW15] ont prouvé que la loi de $Z(\beta)$ satisfait un principe de grandes déviations (PGD) avec un taux β^2 et une certaine fonction de taux $\mathscr{I}$. Comme l'opérateur $\mathcal{H}_L(\beta) = -\Delta + \beta\xi$ est une fonction continue de $Z(\beta)$ (en raison de la continuité des applications (b) et (c) dans la Figure I.2), par le principe de contraction en théorie des grandes déviations, nous sommes en mesure de déduire une estimée de grandes déviations qui implique essentiellement que $\lambda_{n,L}$ "vérifie presque" un PGD lorsque $L \to \infty$. Cela conduit finalement à l'asymptotique (B.6) et à l'identification du préfacteur C_d.

Chapitre III : Spectre de l'Hamiltonien d'Anderson continu sur l'espace plein

Ce chapitre est basé sur un travail en collaboration avec C. Labbé [HL24], soumis à *Probability and Mathematical Physics.*

Nous étudions l'Hamiltonien d'Anderson continu $\mathcal{H}$ sur $L^2(\mathbb{R}^d)$, c'est-à-dire avec $\mathbf{A} = 0$ et $V = \xi$ dans (B.1). La construction d'un tel opérateur sur tout $\mathbb{R}^d$ est plus difficile que dans le cas de volume fini sur $(-L/2, L/2)^d$ car $\mathcal{H}$ n'est plus semi-borné : en effet, les asymptotiques des plus petites valeurs propres dans la limite de volume infini indiquent que le bas du spectre de l'opérateur en volume infini devrait être $-\infty$. Par conséquent, les idées de formulation faible et de formulation mild présentées dans la section I.3.2 ne s'appliquent pas (puisque la forme quadratique n'est pas bornée inférieurement, et on ne s'attend pas à trouver un a suffisamment grand pour lequel (I.11) est solvable). Voir aussi la section I.3.3 pour une discussion pertinente.

Nous suivons plutôt l'idée de la "Construction III" de la section I.3.2. Rappelons que le point de départ est de construire le semi-groupe (c'est-à-dire la solution au problème de Cauchy pour l'équation parabolique), puis d'en extraire son générateur. Il existe plusieurs difficultés techniques : la résolution de cette EDP parabolique n'est pas standard puisque le potentiel est rugueux ; en plus, dans notre contexte le semi-groupe n'est pas composé des opérateurs bornés, l'extraction du générateur à partir du semi-groupe n'est donc pas immédiate.

Notre idée est de résoudre le problème de Cauchy

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x) - \xi(x)u(t,x), & t > 0, x \in \mathbb{R}^d \\ u(t,0) = g \in L^2, & t = 0 \end{cases} \tag{B.7}$$

et montre que la solution vérifie l'hypothèse du Théorème général III.3, ce qui permet d'extraire le générateur $\mathcal{H}$ et donc de donner une définition désirée pour l'Hamiltonien d'Anderson. En effet, la théorie des solutions à (I.48) étant disponible dans [HL15] pour la dimension 2 et dans [HL18] pour la dimension 3, nous devons essentiellement adapter leurs arguments en L^2 et prouver la continuité de l'application $t \mapsto u(t, \cdot)$ en tant que fonction à valeurs dans un espace des fonctions approprié.

De plus, comme le bruit blanc ξ est ergodique par rapport à la translation dans $\mathbb{R}^d$, on peut montrer que l'opérateur auto-adjoint $\mathcal{H}$ ainsi construit a un spectre déterministe (Proposition I.21), c'est-à-dire qu'il existe un ensemble déterministe $\Sigma \subset \mathbb{R}$ tel que $\sigma(\mathcal{H}) = \Sigma$ presque sûrement. La deuxième partie de ce chapitre concerne l'identification de Σ, que nous montrons égal à l'ensemble des réels $\mathbb{R}$. Nous résumons le résultat ci-dessous :

Theorem B.2 (Théorèmes III.1 et III.2)**.** *Pour $d = 2, 3$, l'Hamiltonien d'Anderson continu $\mathcal{H}$ est bien défini sur un domaine dense $\mathcal{D}(\mathcal{H}) \subset L^2(\mathbb{R}^d)$. De plus, $\mathcal{H}$ est ergodique et son spectre coïncide avec $\mathbb{R}$ presque sûrement.*

Le résultat ci-dessus en dimension $d = 2$ a été obtenu récemment par Ueki [Uek23] en utilisant la théorie sophistiquée des distributions paracontrôlées basée sur les noyaux de la chaleur ; notre argument pour $d = 2$ semble être beaucoup plus élémentaire. En revanche, le résultat pour la dimension $d = 3$ est nouveau.

Les principales heuristiques pour montrer $\Sigma = \mathbb{R}$ proviennent de l'observation suivante dûe à Kotani [Kot85], voir aussi [CL90, Corollary V.2.3.] :
Soit Ω un espace de probabilité sur lequel $\mathcal{H} : \xi \mapsto \mathcal{H}^\xi$ est une variable aléatoire à valeurs

dans l'espace des opérateurs auto-adjoints. Fixons deux mesures de probabilité $\mathbb{P}_1, \mathbb{P}_2$ sur Ω pour lesquelles $\mathcal{H}$ est ergodique, et soit Σ_1, Σ_2 le spectre déterministe de $\mathcal{H}$ sous $\mathbb{P}_1$, $\mathbb{P}_2$ respectivement. Supposons que Ω est un espace polonais sur lequel la propriété suivante de continuité concernant la mesure spectrale est vraie : pour tout $\varphi \in L^2$

$$\xi_n \to \xi \text{ in } \Omega \implies \left\langle E^{\xi_n}(\cdot)\varphi, \varphi \right\rangle \text{ converge faiblement vers } \left\langle E^{\xi}(\cdot)\varphi, \varphi \right\rangle, \tag{B.8}$$

où E^{ξ_n}, E^{ξ} désignent la projection-valued measure de l'opérateur $\mathcal{H}^{\xi_n}$, $\mathcal{H}^{\xi}$, respectivement. Alors on a

$$\operatorname{supp} \mathbb{P}_1 \subset \operatorname{supp} \mathbb{P}_2 \implies \Sigma_1 \subset \Sigma_2.$$

En utilisant cette observation, l'idée est de définir $\mathbb{P}_2$ comme la loi du bruit blanc gaussien et $\mathbb{P}_1$ comme la loi du bruit régularisé ξ_ε, pour tout $\varepsilon > 0$ fixé. Évidemment, on a $\operatorname{supp} \mathbb{P}_1 \subset \operatorname{supp} \mathbb{P}_2$ et il n'est pas difficile de montrer que $\Sigma_1 = \mathbb{R}$, ce qui impliquerait $\Sigma_2 = \mathbb{R}$ si (I.49) était vraie. Malheureusement, la propriété (B.8) échoue dans le cas singulier du bruit blanc en raison du fait que l'application (a) dans la Figure B.1 n'est que mesurable.

Pour restaurer la continuité, nous nous concentrons plutôt sur l'espace des modèles admissibles $\mathscr{M}$. Soient $Z_\varepsilon, Z \in \mathscr{M}$ les modèles associés à ξ_ε, ξ, respectivement. Si nous pouvons montrer que le support des lois de Z_ε et Z vérifient

$$\operatorname{supp} Z_\varepsilon \subset \operatorname{supp} Z, \tag{B.9}$$

alors on peut déduire $\Sigma \supset \Sigma_\varepsilon = \mathbb{R}$ du théorème de Kotani, puisque la projection-valued measure est continue par rapport au modèle (applications (b) et (c) dans la Figure B.1).

Dans le Chapitre III, nous donnons un argument complet pour (B.9) en dimension 2 et nous nous appuyons sur le résultat général de Hairer et Schönbauer [HS21] pour la dimension 3.

Chapitre IV : Hamiltonien de Landau avec un potentiel de bruit blanc sur tout l'espace $\mathbb{R}^2$

Ce chapitre est basé sur un travail indépendant en cours.

Nous étudions une construction (sans utiliser une théorie générale des EDP singulières) de l'Hamiltonien de Landau $\mathcal{H}_{\mathbf{A}}$ sur $L^2(\mathbb{R}^2)$ avec un champ magnétique uniforme et perturbé par un potentiel de bruit blanc. C'est-à-dire, $\mathcal{H}_{\mathbf{A}}$ prend la forme (B.1) avec $\mathbf{A}(x, y) = \frac{B}{2}(-y, x)$, $B > 0$ et $V = \xi$. L'argument clé est d'utiliser le théorème des commutateurs de Faris-Lavine (Proposition IV.16), une idée originellement de Ugurcan [Ugu22] concernant l'Hamiltonien d'Anderson continu.

Le résultat souhaité est le suivant :

Theorem B.3 (Theorem IV.1)**.** *L'Hamiltonien de Landau $\mathcal{H}_{\mathbf{A}}$ avec un champ magnétique uniforme perturbé par un potentiel de bruit blanc sur $L^2(\mathbb{R}^2)$ est bien défini et auto-adjoint sur un domaine dense.*

L'idée de la preuve est de décomposer le bruit blanc en la somme

$$\xi = \xi_- + \xi_+$$

de la partie singulière ξ_-, qui est *globalement* $(-1)^-$-Hölder p.s., et de la partie non-bornée ξ_+, qui est continue et satisfait $\xi_+(x) \geq -C(1+|x|^2)$ pour une constante $0 < C < \infty$ p.s.. Avec cette décomposition, nous pouvons raisonner en deux étapes :

1. Premièrement, construire l'opérateur singulier $T = (i\nabla + \mathbf{A})^2 + \xi_-$. Ici, la formulation faible (section I.3.2, Construction II) combinée avec l'astuce de transformation exponentielle introduite dans [HL15] permet de réaliser T comme un opérateur auto-adjoint aléatoire.

2. Deuxièmement, définir $\mathcal{H}_\mathbf{A} = T + \xi_+$ et utiliser le Théorème de Faris-Lavine IV.16 pour montrer que $\mathcal{H}_\mathbf{A}$ est essentiellement auto-adjoint sur le domaine $\mathscr{C} = \{u \in \mathcal{D}(T) : |x|^2 u \in L^2\}$.

Cependant, en rédigeant le Chapitre IV, nous avons réalisé qu'il y a une lacune dans la deuxième étape, déjà présente dans l'article original de Ugurcan [Ugu22]. En effet, pour appliquer Faris-Lavine, on définit l'opérateur auxiliaire

$$N = T + \xi_+ + C(1+|x|)^2.$$

L'objectif est de montrer que N est un opérateur positif, (essentiellement) auto-adjoint sur $\mathscr{C}$ tel que l'estimation du commutateur est vérifiée :

$$\pm i[\mathcal{H}_\mathbf{A}, N] \lesssim N$$

(interprété au sens de forme quadratique sur $\mathscr{C}$). [Ugu22] a argumenté que N est auto-adjoint sur $\mathscr{C}$ et que l'estimation du commutateur est vérifiée, donc H est essentiellement auto-adjoint. La lacune réside dans le raisonnement pour l'auto-adjonction de N : Ugurcan a montré que N est symétrique, fermé et positif sur $\mathscr{C}$, et a conclu que N a une valeur de résolvante réelle et est donc auto-adjoint – cette conclusion est fallacieuse car ces propriétés (symétrique, fermé et positif) n'impliquent pas l'existence d'une valeur de résolvante. Un contre-exemple classique est donné par le Laplacien unidimensionnel sur l'intervalle $(0,1)$ avec le domaine $H_0^2((0,1))$: cet opérateur est symétrique, fermé et positif ; cependant son spectre est tout le plan complexe $\mathbb{C}$.

En effet, la symétrie et la positivité de N impliquent l'existence d'une extension auto-adjointe par le théorème de Friedrichs I.13. Cependant, cette existence ne coïncide pas nécessairement avec la fermeture de N sur $\mathscr{C}$; en fait, nous n'avons a priori aucun contrôle sur le domaine de l'extension de Friedrichs et il pourrait être bien plus large que l'ensemble original $\mathscr{C}$. Cela fait échouer les hypothèses du théorème de Faris-Lavine.

Malgré la lacune dans la construction de l'opérateur $\mathcal{H}_\mathbf{A}$ sur l'espace complet, nous obtenons quelques résultats partiels concernant l'opérateur $\mathcal{H}_{\mathbf{A},L}$ construit sur la boîte bornée $(-L/2, L/2)^d$.

Theorem B.4 (Theorem IV.2)**.** *L'Hamiltonien de Landau $\mathcal{H}_{\mathbf{A},L}$ avec un champ magnétique uniforme perturbé par un potentiel de bruit blanc gaussien sur $(-L/2, L/2)^2$ avec condition de bords de Dirichlet admet une résolvante compacte et ses valeurs propres admettent les mêmes asymptotiques que* (I.47) *en $d = 2$.*

Mentionnons enfin qu'il est possible de construire $\mathcal{H}_{\mathbf{A}}$ en utilisant le théorème de Klein-Landau comme au Chapitre III, mais cela nécessite la théorie des structures de régularité. Nous ne présentons pas les détails mais nous avons l'intention de l'implémenter dans le futur.

Appendice : convergence de résolvante-forte du volume fini à l'infini

Nous incluons un nouveau résultat simple dans l'annexe, abordant la convergence des Hamiltoniens d'Anderson définis sur un volume fini vers ceux définis sur l'espace complet, une notion parfois appelée *limite thermodynamique*. Nous considérons les dimensions $d \in 2, 3$.

Theorem B.5 (Theorem A.1)**.** *Soit $\mathcal{H}$ l'Hamiltonien d'Anderson continu sur $L^2(\mathbb{R}^d)$ et soit $\mathcal{H}_L$ son homologue sur $L^2((-L/2, L/2)^d)$. Alors, $\mathcal{H}_L$ converge vers $\mathcal{H}$ au sens de la résolvante forte lorsque $L \to \infty$, en probabilité.*

B.2 CHAPITRE II : ASYMPTOTIQUE DES PLUS PETITES VALEURS PROPRES DE L'HAMILTONIEN D'ANDERSON CONTINU EN $d \leq 3$

Étant donné un bruit blanc ξ sur $\mathbb{R}^d$, nous considérons le Hamiltonien d'Anderson continu tronqué

$$\mathcal{H}_L := -\Delta + \xi, \quad \text{on } Q_L := (-L/2, L/2)^d,$$

où Δ est le Laplacien continu, les conditions aux limites sont prises comme Dirichlet homogènes et la dimension d est soit 1, 2 ou 3.

Cet opérateur appartient à la classe des opérateurs de Schrödinger aléatoires. La particularité du cadre présent est la singularité du potentiel de bruit blanc, qui est seulement une distribution. En dimension 1, l'opérateur $\mathcal{H}_L$ peut être défini avec des outils standards et des résultats assez complets sont disponibles sur le comportement asymptotique des valeurs propres et des fonctions propres lorsque $L \to \infty$, voir [McK94, DL20, DL23, DL24b].

En revanche, la simple définition de l'opérateur en dimension 2 et 3 n'est pas a priori claire. En effet, la régularité du bruit blanc est trop faible pour que l'opérateur soit défini par des arguments classiques, et il doit en fait être *renormalisé* par des constantes infinies. De nouvelles techniques dans le domaine des EDP stochastiques ont fourni les outils appropriés pour effectuer une telle construction. En s'appuyant sur le calcul paracontrôlé de Gubinelli, Imkeller et Perkowski [GIP15], Allez et Chouk [AC15] ont construit $\mathcal{H}_L$ en dimension 2 sous la condition aux limites périodique. Cette construction a été étendue en dimension 3 sous la condition aux limites périodique par [GUZ20], sous la condition aux limites de Dirichlet en dimension 2 par [Cv21] et aux variétés bidimensionnelles par [Mou21]. D'autre part, une construction sous la condition aux limites périodique et de Dirichlet et pour toute dimension $d \leq 3$ a été présentée dans [Lab19] en utilisant la théorie des structures de régularité [Hai14] : dans le présent article, nous nous appuyons sur cette construction par commodité.

Offrons une brève description de la procédure de renormalisation mentionnée ci-dessus. Considérons l'opérateur $\mathcal{H}_{\varepsilon,L} = -\Delta + \xi_\varepsilon + C_\varepsilon$ associé à un bruit régularisé $\xi_\varepsilon = \xi * \rho_\varepsilon$, où ρ_ε est une fonction lisse à l'échelle ε. Cet opérateur est bien défini puisque ξ_ε est une fonction lisse. Dans les références ci-dessus, il est montré que si l'on choisit correctement la *constante de renormalisation* C_ε, alors $\mathcal{H}_{\varepsilon,L}$ converge au sens de résolvante-norme vers une limite que nous appelons $\mathcal{H}_L$. Noter que, lorsque $\varepsilon \downarrow 0$, C_ε diverge logarithmiquement en dimension 2 et polynomialement en dimension 3. Nous renvoyons le lecteur à la section II.3 pour plus de détails.

In fine, ces constructions produisent un opérateur auto-adjoint $\mathcal{H}_L$ sur $L^2((-L/2, L/2)^d)$ avec un spectre purement ponctuel borné inférieurement : nous notons $(\lambda_{k,L})_{k\geq 1}$ ses valeurs propres dans un ordre croissant et $(\varphi_{k,L})_{k\geq 1}$ les fonctions propres correspondantes normalisées dans L^2. Contrairement au cas de la dimension 1, très peu de choses sont connues sur le spectre de $\mathcal{H}_L$: en dimension 2, le comportement asymptotique des plus petites valeurs propres lorsque $L \to \infty$ a été dérivé dans [Cv21] tandis que l'existence d'une densité d'états a été prouvée dans [Mat21] ; en dimension 3, aucun résultat sur le spectre n'est disponible.

Pour référence ultérieure, rappelons l'inégalité de Gagliardo-Nirenberg - également appelée inégalité de Ladyzhenskaya en dimension 2

$$\|f\|_{L^4(\mathbb{R}^d)} \leq C\|\nabla f\|_{L^2(\mathbb{R}^d)}^{d/4}\|f\|_{L^2(\mathbb{R}^d)}^{1-d/4} , \tag{B.10}$$

et notons κ_d la constante optimale associée, c'est-à-dire

$$\kappa_d := \sup_{f\in H^1(\mathbb{R}^d)} \frac{\|f\|_{L^4(\mathbb{R}^d)}}{\|\nabla f\|_{L^2(\mathbb{R}^d)}^{d/4}\|f\|_{L^2(\mathbb{R}^d)}^{1-d/4}} . \tag{B.11}$$

Le principal résultat de cet article est le suivant.

Theorem B.6. *Fixons $d \in 1, 2, 3$ et $n \in \mathbb{N}$. Alors presque sûrement*

$$\lambda_{n,L} \sim -\left(C_d \log L\right)^{\frac{1}{2-\frac{d}{2}}} , \quad L \to \infty. \tag{B.12}$$

La constante C_d peut être exprimée en termes de la constante de Gagliardo-Nirenberg par la relation

$$C_d = \frac{d^{1+\frac{d}{2}}(4-d)^{2-\frac{d}{2}}}{8}\kappa_d^4 . \tag{B.13}$$

Faisons quelques commentaires sur ce résultat. En dimension 1, la constante de Gagliardo-Nirenberg est connue pour être $\kappa_1 = 3^{-1/8}$ et le résultat se réécrit

$$\lambda_{n,L} \sim -\left(\frac{3}{8}\log L\right)^{2/3} \quad , \quad L \to \infty \, .$$

Cette asymptotique est déjà couverte par des résultats plus précis [McK94, DL20], dans lesquels non seulement le comportement asymptotique de $\lambda_{n,L}$ mais aussi ses fluctuations sont dérivés, voir la fin de l'introduction pour plus de détails. Cela peut également être relié à un résultat de Chen [Che10] sur la masse totale du modèle d'Anderson parabolique associé.

En dimension supérieure, la constante de Gagliardo-Nirenberg κ_d n'est plus explicite. En dimension 2, l'asymptotique est

$$\lambda_{n,L} \sim -\kappa_2^4 \log L \, , \quad L \to \infty \, ,$$

et a été récemment établie par Chouk et van Zuijlen [Cv21] (voir aussi [GL22] pour des résultats connexes sur les bruits gaussiens réguliers). Une amélioration mineure par rapport à leur résultat est que notre convergence a lieu presque sûrement lorsque $L \to \infty$, et pas seulement sur des sous-suites $L_k \to \infty$, voir la Remarque II.4 pour quelques explications. En dimension 3, l'asymptotique est

$$\lambda_{n,L} \sim -\frac{243}{64}\kappa_3^8 \, (\log L)^2 \quad , \quad L \to \infty \, ,$$

et dans ce cas, le résultat est nouveau.

Notre preuve est réalisée simultanément dans toutes les dimensions $d \leq 3$ afin de souligner la dépendance en d des arguments et du résultat global. Soulignons que nous suivons la même stratégie de preuve que Chouk et van Zuijlen [Cv21] qui ont couvert le cas de la dimension 2. En fait, la preuve se résume essentiellement à établir une estimation de queue sur la distribution de probabilité de la valeur propre principale : c'est le contenu du résultat suivant.

Theorem B.7. *Fixons $\eta \in (0,1)$ et $n \geq 1$. Il existe $\gamma_2 > \gamma_1 > 0$ et $x_0 > 0$ tels que les inégalités suivantes soient vérifiées : pour tout $L \geq 1$ et tout $x \geq x_0$, nous avons*

$$\exp\left[-\gamma_2 x^{\frac{d}{2}} e^{d\log L - (1-\eta)\rho x^{2-\frac{d}{2}}}\right] \leq \mathbb{P}(\lambda_{n,L} \geq -x) \leq \exp\left[-\gamma_1 x^{\frac{d}{2}} e^{d\log L - (1+\eta)\rho x^{2-\frac{d}{2}}}\right] \quad \text{(B.14)}$$

avec $\rho = d/C_d$.

Observons que dans la limite $L \to \infty$ et pour un petit η (prenons $\eta = 0$ pour simplifier), les fonctions les plus à gauche et les plus à droite dans (B.14) passent abruptement de 0 à 1 autour de la valeur critique $x_c = (C_d \log L)^{1/(2-d/2)}$ où l'exposant $d\log L - \rho x^{2-d/2}$ s'annule. Cela implique que la fonction de répartition $x \mapsto \mathbb{P}(\lambda_{n,L} \geq -x)$ est proche de 0 pour $x \ll x_c$ et proche de 1 pour $x \gg x_c$, et donc que la distribution de $-\lambda_{n,L}$

se concentre près de cette valeur critique. Étant donné ce résultat, la dérivation du Théorème II.1 est relativement élémentaire.

Nous concluons cette introduction avec quelques conjectures. Soit a_L la solution unique de l'équation

$$\frac{d}{2}\log a_L + d\log L - \rho a_L^{2-d/2} = 0 \ ,$$

et définissons

$$b_L := \frac{C_d}{d(2-\frac{d}{2})a_L^{1-\frac{d}{2}}} \ .$$

Notons que le développement asymptotique de a_L est donné par

$$a_L^{2-d/2} := (C_d \log L)\left[1 + \frac{1}{4-d}\frac{\log\log L}{\log L} + o\left(\frac{\log\log L}{\log L}\right)\right] . \tag{B.15}$$

Conjecture 3. *Prenons $d \in \{1,2,3\}$. Le processus ponctuel $\left(\frac{\lambda_{n,L}+a_L}{b_L}\right)_{n\geq 1}$ converge en loi lorsque $L \to \infty$ vers un processus ponctuel de Poisson sur $\mathbb{R}$ d'intensité $e^x dx$. En particulier, la v.a.*

$$-\frac{\lambda_{1,L}+a_L}{b_L}$$

converge en loi vers une variable aléatoire de Gumbel.

Notre deuxième conjecture concerne le comportement asymptotique des fonctions propres près de leurs maxima. Nous notons $U_{n,L} \in [-L/2, L/2]^d$ le point où $|\varphi_{n,L}|$ atteint son maximum global. Soit Q l'unique solution radiale positive sur $\mathbb{R}^d$ de

$$-\Delta Q - Q^3 = -Q \ .$$

Il est connu que – à des translations, dilatations et mises à l'échelle près – Q est l'unique optimiseur de l'inégalité de Gagliardo-Nirenberg (II.1), voir [Fra14] et les références qui y sont citées. On peut déduire de [Lew10, Sec.5] que $\|Q\|_{L^4}^4 = \frac{2d}{C_d}$.

Conjecture 4. *Prenons $d \in \{1,2,3\}$. Pour tout $n \geq 1$, la convergence suivante a lieu en probabilité lorsque $L \to \infty$*

$$\left(\frac{1}{a_L^{d/4}}|\varphi_{n,L}|\Big(U_{n,L} + \frac{x}{\sqrt{a_L}}\Big), x \in \mathbb{R}^d\right) \Rightarrow \psi_* \ ,$$
$$\left(\frac{1}{a_L}\xi\Big(U_{n,L} + \frac{x}{\sqrt{a_L}}\Big), x \in \mathbb{R}^d\right) \Rightarrow -\frac{{\psi_*}^2}{\|\psi_*\|_{L^4(\mathbb{R}^d)}^2}\sqrt{\frac{2d}{C_d}} \ ,$$

avec $\psi_*(x) = Q(x)/\|Q\|_{L^2}$.

Dans la Conjecture 4, la première convergence a lieu dans un espace de distributions ; l'abus de notation concernant la mise à l'échelle sur ξ doit être interprété au sens distributionnel, c'est-à-dire en passant les opérations de mise à l'échelle aux fonctions de test.
En dimension 1, ces deux conjectures ont été en fait prouvées par Dumaz et Labbé [DL20] (la convergence vers une v.a. de Gumbel a été prouvée plus tôt par McKean [McK94]). Dans ce cas, nous avons

$$a_L \sim \left(\frac{3}{8}\log L\right)^{2/3} , \quad b_L = \frac{1}{4\sqrt{a_L}} , \quad \psi_* = \frac{1}{\sqrt{2}\cosh} , \quad \frac{{\psi_*}^2}{\|\psi_*\|^2_{L^4(\mathbb{R}^d)}}\sqrt{\frac{2d}{C_d}} = \frac{2}{\cosh^2} .$$

B.3 Chapitre III : Construction et spectre de l'Hamiltonien d'Anderson continue sur $\mathbb{R}^2$ et $\mathbb{R}^3$

Cet article établit la construction rigoureuse et certaines propriétés spectrales de l'opérateur de Schrödinger aléatoire

$$\mathcal{H} := -\Delta + \xi , \quad \text{sur } \mathbb{R}^d ,$$

où $d \in \{2, 3\}$, Δ est le Laplacien continu et ξ est un bruit blanc sur $\mathbb{R}^d$.

Rappelons que les propriétés spectrales des opérateurs de Schrödinger aléatoires $-\Delta + V$ ont attiré beaucoup d'attention dans la communauté de la physique mathématique, notamment dans le cas discret où Δ est le Laplacien sur $\mathbb{Z}^d$ et $V(k), k \in \mathbb{Z}^d$ est une collection de variables aléatoires i.i.d. En particulier, le phénomène de localisation d'Anderson [And58] a donné lieu à une littérature abondante. Le potentiel considéré dans cet article provient de la limite d'échelle d'une large classe de champs aléatoires discrets ou continus, motivant ainsi l'étude de l'opérateur $\mathcal{H}$. D'un point de vue physique, cet opérateur peut être vu comme un modèle idéalisé dans une situation où le potentiel V a une longueur de corrélation très petite.

Étant donné que le bruit blanc est à valeurs dans l'espace des distributions, la simple définition de l'opérateur $\mathcal{H}$ est délicate. En réalité, l'opérateur nécessite une renormalisation par des constantes infinies et des avancées récentes [GIP15, Hai14] dans le domaine des EDP stochastiques fournissent les outils nécessaires pour accomplir cette tâche. Explorons brièvement la procédure de renormalisation. On considère le bruit lissé $\xi_\varepsilon := \xi * \varrho_\varepsilon$ où $\varrho_\varepsilon(x) := \varepsilon^{-d}\varrho(x/\varepsilon)$, $x \in \mathbb{R}^d$, pour une fonction paire et régulière ϱ à support compact dans la boule unité de $\mathbb{R}^d$ et intégrant à 1. Si on remplace ξ par ξ_ε, alors la définition de l'opérateur rentre dans le cadre des résultats classiques [Kat72, FL74]. Cependant, la suite d'opérateurs ne converge pas lorsque $\varepsilon \downarrow 0$. Au lieu de cela, il faut identifier soigneusement une constante divergente (avec ε) C_ε telle que la collection d'opérateurs auto-adjoints

$$\mathcal{H}_\varepsilon := -\Delta + \xi_\varepsilon + C_\varepsilon ,$$

converge vers une limite que nous appelons $\mathcal{H}$. Notons que C_ε diverge comme $(2\pi)^{-1}\log\varepsilon^{-1}$ en dimension 2, tandis qu'en dimension 3, il diverge comme $c_\varrho\varepsilon^{-1}+\ln\varepsilon^{-1}$ pour une constante $c_\varrho>0$.

Ce programme a été réalisé en volume fini (un tore, une boîte bornée) en dimension 2 et 3 par plusieurs auteurs [AC15, GUZ20, Lab19, Cv21, Mou21, Mv22, BDM23]. La convergence de $\mathcal{H}_\varepsilon$ vers $\mathcal{H}$ est alors obtenue dans le sens de la résolvante, en probabilité. Soulignons qu'en volume fini, l'opérateur est borné inférieurement de sorte que son ensemble résolvant admet des valeurs réelles et qu'un argument de point fixe peut être appliqué pour construire les opérateurs résolvants associés.
Il s'avère que la construction en volume infini est beaucoup plus difficile. La principale raison en est que l'opérateur n'est plus borné inférieurement (voir par exemple les asymptotiques de grand volume sur l'état fondamental [Cv21, HL22]), de sorte que la stratégie de preuve ci-dessus n'est plus applicable.
Dans un travail récent, Ugurcan [Ugu22] a proposé une construction de $\mathcal{H}$ sur $\mathbb{R}^2$ en adaptant un critère célèbre [FL74] de Faris et Lavine assurant l'auto-adjonction essentielle pour les opérateurs de Schrödinger. Notons cependant qu'Ugurcan ne traite pas de la convergence de $\mathcal{H}_\varepsilon$ vers $\mathcal{H}$. Plus récemment, Ueki [Uek23] a proposé une autre construction de $\mathcal{H}$ sur $\mathbb{R}^2$, basée sur l'approche du semi-groupe de la chaleur de [Mou21] : il est démontré que $\mathcal{H}_\varepsilon$ converge dans le sens de la résolvante-forte vers $\mathcal{H}$ et que le spectre de $\mathcal{H}$ est presque sûrement égal à $\mathbb{R}$.
D'autre part, la construction de $\mathcal{H}$ sur $\mathbb{R}^3$ n'a pas encore été abordée jusqu'à présent.

Les deux travaux mentionnés ci-dessus parviennent à identifier le domaine de l'opérateur aléatoire $\mathcal{H}$ sur $\mathbb{R}^2$ en utilisant le calcul paracontrôlé. Notons que ces travaux sont techniquement assez complexes. Dans cet article, nous présentons une construction relativement simple de $\mathcal{H}$, dans les dimensions 2 et 3, basée sur la théorie des solutions du modèle parabolique d'Anderson (PAM)

$$\begin{cases}\partial_t u = \Delta u - u\xi\ , & \text{sur } \mathbb{R}^d\ ,\\ u(t=0,\cdot)=f(\cdot)\ . \end{cases}$$

La construction de cette EDP est a priori moins délicate que la construction de l'opérateur auto-adjoint $\mathcal{H}$: en effet, la théorie des solutions pour les EDP est flexible puisqu'il suffit d'identifier un espace (raisonnable) dans lequel l'existence et l'unicité sont vérifiées, tandis que l'auto-adjonction est une propriété délicate qui nécessite d'identifier un sous-ensemble dense (et aléatoire) de L^2 sur lequel l'opérateur agit. Soulignons que la construction du (PAM) a été établie dans [HL15, HL18] respectivement sur $\mathbb{R}^2$ et $\mathbb{R}^3$.
Nous définissons l'opérateur $\mathcal{H}$ comme le générateur du semi-groupe associé à la collection de solutions de cette EDP. Cela constitue le contenu de notre premier résultat :

Theorem B.8. *En dimensions 2 et 3, il existe un opérateur aléatoire $\mathcal{H}$ qui est auto-adjoint sur (un sous-ensemble dense de) $L^2(\mathbb{R}^d,dx)$ et qui est la limite en probabilité de $\mathcal{H}_\epsilon$ lorsque $\epsilon\downarrow 0$ dans le sens de la résolvante forte. Pour tout $t\geq 0$, le domaine de* *l'opérateur* $e^{-t\mathcal{H}}$ *contient toutes les fonctions* $f \in L^2(\mathbb{R}^d, dx)$ *à support compact et* $e^{-t\mathcal{H}}f$ *coïncide avec la solution de* (III.1) *au temps* t*, en partant de* f *au temps* 0*.*

La preuve du Théorème B.8 est divisée en trois étapes :

$$\begin{array}{ccccccc} \text{Bruit} & \xrightarrow{(1)} & \text{Bruit augmenté} & \xrightarrow{(2)} & \text{(PAM)} & \xrightarrow{(3)} & \text{Générateur} \\ \xi & & Q(\xi) & & u & & \mathcal{H} \end{array}$$

L'étape (1) est standard dans le domaine des EDP stochastiques singulières : elle associe de manière mesurable (mais typiquement non continue) un bruit augmenté à un bruit (typiquement rugueux). Le point clé est que les données supplémentaires contenues dans le bruit augmenté permettent de récupérer la continuité dans les étapes suivantes. Notez que la renormalisation est mise en œuvre à cette étape.

L'étape (2) est déterministe : étant donné le bruit augmenté $Q(\xi)$, on définit une théorie de solution pour le (PAM). En dimension 2, cela repose sur des arguments relativement élémentaires que nous rappelons dans la section III.3. En revanche, la construction du (PAM) en dimension 3 est complexe et repose sur la théorie des structures de régularité [Hai14] : nous utiliserons les résultats de [HL18], voir section III.4.

Si on note par $u^f(t,\cdot)$ la solution de (PAM), alors $P_t f := u^f(t,\cdot)$ est un semi-groupe. L'étape (3) construit (de manière déterministe) un opérateur auto-adjoint unique $\mathcal{H}$ satisfaisant $P_t f = e^{-t\mathcal{H}}f$. Cependant, extraire $\mathcal{H}$ de ce semi-groupe est beaucoup plus subtil qu'on ne le pense.
Une tentative naturelle consisterait à construire la résolvante de $\mathcal{H}$ par intégration dans le temps du semi-groupe : $\int_0^\infty e^{-at}u^f(t,\cdot)dt$. Cependant, l'opérateur n'est pas borné inférieurement, de sorte que rien ne garantit que $(\mathcal{H}+a)^{-1}f$ existe pour des paramètres réels a. De plus, la théorie de solution du (PAM) fournit des bornes a priori très mauvaises sur la croissance en temps de $u^f(t,\cdot)$, de sorte qu'on ne peut pas isoler de bonnes conditions initiales f pour lesquelles l'intégrale en temps ci-dessus converge. En fait, ces fonctions f pour lesquelles l'intégrale en temps converge devraient être des fonctions qui, informellement parlant, “évitent” ces régions de l'espace où ξ est très grand, confirmant ainsi qu'on ne peut pas identifier un ensemble simple de telles fonctions.
Une deuxième approche consisterait à appliquer l'extension de Friedrich aux formes quadratiques $\langle f, u^f(t,\cdot)\rangle$ (pour des fonctions f assez bonnes) et à exploiter la propriété de semi-groupe satisfaite par la solution de (PAM) : nous n'avons pas réussi à conclure avec cette approche. Nous sommes alors tombés sur un (beau) résultat de Klein et Landau [KL81] qui fournit le bon cadre. Cela est présenté dans la section III.2, et les estimations requises sur le (PAM) pour appliquer ce résultat et déduire le Théorème III.1 sont présentées dans les sections III.3 et III.4.

Soulignons que la construction de l'opérateur via son semi-groupe s'accompagne de belles propriétés de continuité, qui permettent, en particulier, de prouver la convergence de la résolvante-forte de l'énoncé, voir les Propositions III.12 et III.28. Nous tenons à souligner que l'approche présentée ici est robuste et peut certainement être appliquée à une large classe d'opérateurs différentiels aléatoires singuliers.

Notre deuxième résultat est l'identification du spectre de $\mathcal{H}$.

Theorem B.9. *En dimensions* 2 *et* 3*, presque sûrement, le spectre de* $\mathcal{H}$ *est* $\mathbb{R}$*.*

Le fait que le spectre soit presque sûrement un ensemble déterministe découle de l'ergodicité du bruit blanc combinée avec la propriété de commutation suivante impliquant le bruit décalé $\theta_x\xi(\cdot) = \xi(\cdot - x)$ et les opérateurs de translation $\mathcal{T}_x f(\cdot) = f(\cdot + x)$ (voir section III.5 pour plus de détails)

$$\mathcal{H}(\theta_x\xi) = \mathcal{T}_x^*\mathcal{H}(\xi)\mathcal{T}_x \ , \quad \forall x \in \mathbb{R}^d \ .$$

Bien que cette propriété soit claire de manière informelle, pour l'établir rigoureusement, il faut construire simultanément les bruits augmentés associés aux bruits décalés $\theta_x\xi$ par tout $x \in \mathbb{R}^d$. C'est l'objectif des Lemmes III.4 et III.21.

Une stratégie habituelle pour montrer qu'une valeur $\lambda \in \mathbb{R}$ appartient au spectre d'un opérateur auto-adjoint A consiste à construire une suite de Weyl, à savoir une suite de fonctions f_n de norme L^2 unitaire telles que $(A - \lambda)f_n$ converge vers 0 dans L^2. Cet argument est mis en œuvre pour l'opérateur $\mathcal{H}_\varepsilon$ dans la Proposition III.31, et permet de montrer que le spectre de ce dernier est tout $\mathbb{R}$ presque sûrement. Pour l'opérateur limite $\mathcal{H}$, cependant, cet argument est délicat à mettre en œuvre car le domaine de l'opérateur est composé de fonctions non lisses qui dépendent de manière subtile du bruit augmenté. Soulignons qu'Ueki [Uek23] a réussi à implémenter des suites de Weyl en dimension 2 en utilisant des arguments d'approximation délicats, puis a montré que le spectre est tout $\mathbb{R}$ presque sûrement.

Dans le présent travail, nous nous appuyons sur un autre argument qui fonctionne à la fois en dimension 2 et 3. L'idée sous-jacente provient d'un résultat de Kotani [Kot85] qui peut être formulé comme suit : si on montre que le support (topologique) de ξ_ε est inclus dans le support (topologique) de ξ, alors le spectre de $\mathcal{H}_\varepsilon$ est inclus dans le spectre de $\mathcal{H}$. Comme la Proposition III.31 montre que le premier est $\mathbb{R}$, nous en déduisons le résultat souhaité.

Bien que cette idée informelle soit rigoureuse pour les potentiels à valeurs fonctionnelles comme démontré dans [Kot85], dans le contexte singulier considéré dans cet article, elle n'est que heuristique. En effet, l'argument de Kotani repose sur la continuité de l'opérateur par rapport au bruit : dans la situation présente, l'opérateur n'est mesurable qu'avec le bruit mais est continu avec le bruit augmenté. Par conséquent, nous devons montrer que le support (topologique) du bruit augmenté associé à ξ_ε est inclus dans le support (topologique) du bruit augmenté associé à ξ, et nous montrerons dans la section III.5 que nous pouvons étendre l'argument de Kotani à ce cadre.

En dimension 2, nous fournissons une preuve complète de l'inclusion des supports des bruits augmentés, voir la Sous-section III.3.4. En dimension 3, nous nous appuyons sur un résultat général de Hairer et Schönauer [HS21] qui établit un théorème de support pour les EDP stochastiques singulières, voir la Sous-section III.4.4.

B.4 Chapitre IV : Hamiltonien de Landau perturbé par le bruit blanc gaussien sur $\mathbb{R}^2$ et l'asymptotique en grande volume des plus petites valeurs propres

Dans ce travail, nous considérons l'Hamiltonien de Landau en dimension 2 perturbé par un bruit blanc gaussien, représenté formellement par

$$\mathcal{H} := (i\nabla + \mathbf{A})^2 + \xi. \tag{B.16}$$

Dans le contexte physique, le champ vectoriel $\mathbf{A}$ représente le potentiel magnétique, c'est-à-dire que $\nabla \times \mathbf{A}$ coïncide avec le champ magnétique externe. Nous nous intéressons au cas où le champ magnétique externe a une magnitude constante $B \geq 0$ et est dirigé le long de l'axe z. Dans ce cas, nous choisissons le *jauge symétrique*, c'est-à-dire que $\mathbf{A}$ est donné par

$$\mathbf{A}(x_1, x_2) = \frac{B}{2}(-x_2, x_1), \quad (x_1, x_2) \in \mathbb{R}^2. \tag{B.17}$$

L'Hamiltonien de Landau est un objet qui apparaît naturellement dans de divers modèles physiques, principalement dans celui de l'effet Hall quantique. Classiquement, l'effet Hall est un phénomène lié à la force de Lorentz induite par le champ magnétique. En effet, une particule chargée se déplaçant dans un champ magnétique uniforme subira une force perpendiculaire au champ et à sa vitesse, effectuant ainsi un mouvement circulaire sur une orbite correspondant à l'énergie du système.

Lorsqu'un courant constant de particules chargées est confiné dans un conducteur surlequel un champ magnétique est appliqué, la force de Lorentz se traduit par un champ électrique induit pointé dans la direction perpendiculaire au courant et au champ magnétique appliqué. En mesurant le rapport entre les magnitudes du courant et du potentiel électrique induit, on définit la conductance de Hall $\sigma_{\mathcal{H}}$. Dans le cadre classique, $\sigma_{\mathcal{H}}$ croît linéairement avec l'inverse de la magnitude magnétique $1/B$. Cependant, dans le régime quantique, les expériences montrent que $\sigma_{\mathcal{H}}$ ne prend que des multiples entiers d'une quantité fixée, d'où l'appellation d'effet Hall quantique. Plus de détails peuvent être trouvés dans l'introduction de [BvS94].

L'effet Hall quantique est expliqué par la quantification de l'énergie totale (Hamiltonien) du système, conduisant à l'opérateur de Landau, ou au Laplacien magnétique

$$(i\nabla + \mathbf{A})^2$$

avec $\mathbf{A}$ donné par (IV.2). En effet, sur tout l'espace $\mathbb{R}^2$, le spectre de l'Hamiltonien de Landau est purement essentiel et composé de valeurs propres données par $(2n+1)B, n = 0, 1, 2, \ldots$, chacune avec une multiplicité infinie. Ces valeurs propres isolées sont appelées les *niveaux de Landau*, et sont étroitement liées au comportement de croissance quantifié de la conductance de Hall $\sigma_{\mathcal{H}}$.

Une considération physiquement pertinente survient lorsqu'il y a des défauts présents dans le conducteur. Dans ce cas, ces défauts sont généralement modélisés par un potentiel aléatoire V pour capturer leur nature non-déterministe, de sorte que la quantification conduit à l'Hamiltonien de la forme

$$(i\nabla + \mathbf{A})^2 + V.$$

Une question naturelle est alors de comprendre l'impact de V sur les propriétés spectrales de l'Hamiltonien. Tout d'abord, on souhaite définir l'Hamiltonien $(i\nabla + \mathbf{A})^2 + V$ en tant qu'opérateur auto-adjoint non borné sur un certain espace de Hilbert. Ensuite, on étudie la nature spectrale d'un tel opérateur, en particulier si l'Hamiltonien présente une localisation spectrale, c'est-à-dire l'existence d'un spectre ponctuel dans certains régimes d'énergie avec des fonctions propres exponentiellement décroissantes.

La classe de potentiels aléatoires la plus étudiée est celle de type alliage. C'est-à-dire que V prend la forme

$$V(x) = \sum_{i \in \mathbb{Z}^d} q_i u(x - i)$$

modélisant les défauts situés aux points du réseau de $\mathbb{Z}^d$. Dans l'expression ci-dessus, u est une fonction déterministe, non négative, continue et à support compact appelée *potentiel à site unique*, modélisant le potentiel introduit par un seul défaut, tandis que (q_i) est une famille i.i.d. de variables aléatoires nommées *constantes de couplage*, chacune suivant une densité de probabilité g. Le plus souvent, on pose une hypothèse sur la densité de probabilité g : lorsque le support de g est compact, [CH96] montre que le spectre de l'Hamiltonien présente une structure de bande, c'est-à-dire que le spectre est contenu dans des bandes de largeur $2\|V\|_\infty$ centrées sur les niveaux de Landau, et de plus le spectre proche des bords de bande est ponctuel avec des fonctions propres exponentiellement décroissantes. Dans le cas où les constantes de couplage, et ainsi le potentiel lui-même, sont non bornées, on s'attend à ce que les niveaux de Landau soient complètement estompés. Par exemple, dans [BCH97], où la densité de probabilité g n'est plus supposée à support compact, les auteurs montrent qu'en vertu de certaines conditions sur les moments de g, l'Hamiltonien est auto-adjoint avec un spectre égal à $\mathbb{R}$, et que les lacunes entre les niveaux de Landau sont partiellement remplies avec des états localisés.

Un autre choix de V qui attire l'attention des physiciens est le potentiel gaussien, c'est-à-dire que V est un champ gaussien centré avec une certaine fonction de covariance $C(x) = \mathbb{E}[V(x)V(0)]$, étant donné que ce modèle ne suppose pas de structure sous-jacente des défauts. Tout comme dans le cas de l'alliage avec des constantes de couplage non bornées, on s'attend à ce que le spectre perturbé soit l'ensemble $\mathbb{R}$ entier du fait que V peut prendre des valeurs arbitrairement grandes avec une probabilité positive. Sous des hypothèses sur la régularité de la fonction de covariance C, [FLM00] montre que c'est effectivement le cas et en outre le spectre est localisé pour des énergies assez négatives. En physique, on considère également la limite lorsque la fonction de covariance $C(x)$ tend vers un delta de Dirac $c\delta(x)$, ce qui correspond à notre cas de bruit blanc gaussien. Bien que l'opérateur avec un potentiel de bruit blanc soit mal défini, Wegner [Weg83] a déjà dérivé une expression exacte pour la *densité d'états restreinte à un certain niveau de Landau*, une quantité qui mesure le nombre d'états par unité de volume et par intervalle d'énergie infinitésimal pour l'Hamiltonien projeté sur un niveau de Landau.

À notre connaissance, il n'existe pas d'étude supplémentaire sur l'Hamiltonien de Landau avec le bruit blanc gaussien dans la communauté mathématique. Le présent travail sert donc d'essai initial pour combler le fossé entre les mathématiques et la physique.

Pour une enquête complète sur les opérateurs de Schrödinger aléatoires, nous renvoyons les lecteurs à [LMW03].

Le premier défi auquel on est confronté dans la définition de $\mathcal{H}$ est la faible régularité du bruit blanc ξ : comme ξ est presque sûrement seulement une distribution, on remarque immédiatement que le domaine de $\mathcal{H}$ ne peut contenir aucune fonction régulière. D'autre part, si f doit être dans le domaine de $\mathcal{H}$, on s'attend à avoir une certaine annulation entre $(i\nabla + \mathbf{A})^2 f$ et $\xi \cdot f$ de sorte que la somme puisse être une fonction de L^2. Comme ξ est localement de régularité de Hölder α p.s. pour tout $\alpha < -d/2$, cela nécessite que les éléments du domaine aient une régularité $2 + \alpha$. Par conséquent, on vérifie que le produit $\xi \cdot f$ n'est bien défini que pour la dimension $d = 1$ et singulier pour $d \geq 2$.

De tels produits mal définis constituent également une difficulté majeure pour l'étude des EDP singulières, pour lesquelles un procédé appelé renormalisation est nécessaire : on mollifie d'abord le bruit par une approximation de l'unité, puis on retire des constantes divergentes choisies de manière appropriée de la solution régularisée avant de prendre la limite. Ce n'est que récemment que des percées majeures ont été obtenues dans ce domaine lorsque Hairer [Hai14] a introduit la théorie des structures de régularité et que Gubinelli, Imkeller et Perkowski [GIP15] ont introduit les distributions paracontrôlées. Ces théories intègrent avec succès l'idée de renormalisation dans la résolution de ces problèmes d'EDP mal posés.

Avec ces nouveaux outils en main, des progrès ont été réalisés dans la construction d'opérateurs de Schrödinger aléatoires impliquant du bruit blanc. Par exemple, [AC15] a défini l'Hamiltonien d'Anderson avec bruit blanc gaussien (c'est-à-dire $\mathbf{A} = 0$) sur le tore bidimensionnel en utilisant la théorie du calcul paracontrôlé. Cette construction est ensuite généralisée à la dimension 3 par [GUZ20]. Par la théorie des structures de régularité, une construction de l'Hamiltonien d'Anderson dans les dimensions générales $d \leq 3$ avec conditions aux bords périodiques ainsi que de Dirichlet est proposée dans [Lab19]. Dans le cas avec champ magnétique, [MM21] a construit l'Hamiltonien de Landau sans potentiel mais avec champ magnétique de bruit blanc.

Dans le présent travail, l'objectif est double : premièrement, nous construisons l'opérateur sur l'espace complet $\mathbb{R}^2$ où les effets du champ magnétique sur le spectre sont réellement visibles ; deuxièmement, nous remplaçons les théories sophistiquées de solutions d'EDP par une technique de conjugaison simple inspirée de [HL15] qui nous épargne des théories de renormalisation compliquées. Nous énonçons donc le principal résultat qui donne la définition de l'Hamiltonien de Landau sur l'espace complet.

Theorem B.10 (*). *L'opérateur non-borné $\mathcal{H}$* (IV.1) *défini au sens de renormalisation est auto-adjoint sur un domaine dense $\mathcal{D}(\mathcal{H})$ contenu dans $L^2(\mathbb{R}^2)$.*

L'obstacle principal pour la construction sur l'espace complet est que le bruit blanc spatial ξ n'est pas *globalement* Hölder sur $\mathbb{R}^2$. Pour faire face à cela, l'argument ici est basé sur l'observation que nous pouvons séparer le problème de la régularité de celui de la non-bornitude par la décomposition $\xi = \xi^- + \xi^+$, où ξ^- est une distribution vivant dans un espace de Hölder global tandis que ξ^+ est une fonction non-bornée dont la croissance peut être contrôlée. Une telle décomposition est déjà notée par Gubinelli et Hofmanová [GH19] en utilisant l'analyse de Fourier. Dans la section IV.3.1, nous proposons une preuve alternative de ce fait en utilisant la décomposition en ondelettes.

Grâce à la décomposition, nous écrivons

$$\mathcal{H} = T + \xi^+, \quad \text{où} \quad T = (i\nabla + \mathbf{A})^2 + \xi^-.$$

Cela nous permet de factoriser la preuve du Théorème IV.1 en deux étapes.
Premièrement, nous traitons la partie irrégulière et construisons l'opérateur T : au lieu d'utiliser des théories d'EDP singulières, nous conjuguons T par un objet stochastique construit à partir du bruit global Hölder ξ^-. En un sens, un tel objet stochastique joue le rôle de *modèles* ou de *bruit augmenté*. L'opérateur conjugué ainsi obtenu $\bar{T}$ a un meilleur comportement en termes de régularité, et nous sommes capables d'étudier sa forme quadratique associée et de définir l'opérateur lui-même par extension de Friedrichs. Nous notons que cette partie de la construction peut être vue comme un cas spécial du travail récent de Matsuda et van Zuijlen [Mv22] où ils ont construit l'Hamiltonien d'Anderson sur un domaine borné perturbé par une variété de bruits en dimension jusqu'à 3.
Deuxièmement, T étant réalisé comme un opérateur auto-adjoint, la définition de $\mathcal{H} = T + \xi^+$ peut être atteinte par un résultat classique dû à Faris et Lavine [FL74], qui nous demande de trouver un opérateur auxiliaire auto-adjoint N et de contrôler le commutateur $[T+\xi^+, N]$. Cette observation est d'abord remarquée par Ugurcan dans [Ugu22] où il a construit l'Hamiltonien d'Anderson avec bruit blanc gaussien dans l'espace complet $\mathbb{R}^2$.

La première étape de la construction ci-dessus peut être appliquée pour définir les opérateurs $\mathcal{H}_L$ sur des cubes ouverts bornés $Q_L = (-L/2, L/2)^2$ avec des conditions aux bords de Dirichlet. Dans ce cas, $\mathcal{H}_L$ admet des résolvantes compactes et donc son spectre est composé de valeurs propres isolées avec une multiplicité finie $\lambda_{1,L} \leq \lambda_{2,L} \leq \dots$. Comme résultat secondaire, nous obtenons l'asymptote presque sûre de $\lambda_{n,L}$ lorsque L tend vers l'infini. Ce résultat est en ligne avec les résultats de l'Hamiltonien d'Anderson prouvés dans [Cv21].

Theorem B.11. *Les valeurs propres de $\mathcal{H}_L = (i\nabla + \mathbf{A})^2 + \xi$ agissant sur $L^2(Q_L)$ satisfont*

$$\lambda_{n,L} \sim -C \log L \text{ lorsque } L \to +\infty \quad p.s.,$$

où C est la constante optimale dans l'inégalité de Ladyzhenskaya : $\|f\|_{L^4} \leq C \|\nabla f\|_{L^2}^{1/2} \|f\|_{L^2}^{1/2}$ pour tout $f \in H^1$.

Le Théorème IV.2 est impliqué par le Théorème IV.28 ci-dessous par un argument de Borel-Cantelli. Ici, nous établissons uniquement la preuve du Théorème IV.28 dans la section IV.4 et renvoyons les lecteurs à la preuve de [HL22, Thm. 1] pour l'étape finale. Bien que la stratégie de preuve ici soit exactement la même que celle de [Cv21] et [HL22], les lemmes intermédiaires deviennent plus transparents et concis puisque notre construction ne nécessite pas de machinerie des structures de régularité ou des distributions paracontrôlées.

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RÉSUMÉ

Cette thèse porte sur les opérateurs de Schrödinger aléatoires dans un cadre continu, en particulier ceux avec un potentiel de bruit blanc gaussien. La définition de ces opérateurs différentiels est généralement non triviale et nécessite la renormalisation dans les dimensions $d \geq 2$. Nous présentons d'abord un cadre général pour traduire le problème de construction de l'opérateur en EDP stochastiques. Cette approche nous permet de définir l'opérateur en question, d'établir son auto-adjonction et d'étudier son spectre.
Par la suite, nous passons à l'étude de l'Hamiltonien d'Anderson continu dans deux configurations spatiales distinctes : d'abord dans une boîte bornée de longueur latérale L avec une condition de bord de Dirichlet nulle pour les dimensions $d \leq 3$, et ensuite dans l'espace Euclidien $\mathbb{R}^d$, pour $d \in \{2, 3\}$. Dans le premier cas, l'opérateur admet des valeurs propres $\lambda_{n,L}$, pour lesquelles nous identifions l'asymptotique presque sûre lorsque $L \to \infty$. Cet asymptotique est conforme aux résultats antérieurs dans la littérature pour les dimensions 1 et 2, tandis que notre résultat en dimension 3 est nouveau. Dans le second cas, nous proposons une nouvelle technique de construction en utilisant la théorie des solutions de l'équation parabolique associée, ce qui permet de prouver l'auto-adjonction et de montrer que le spectre est presque sûrement égal à $\mathbb{R}$. Cette approche confirme le résultat récemment établi en dimension 2 dans la littérature, cependant notre construction semble plus élémentaire ; pour la dimension 3, notre résultat est nouveau.
Enfin, nous présentons un projet en cours qui aborde le cas où un champ magnétique uniforme est appliqué au système : cela conduit à l'étude de l'Hamiltonien de Landau perturbé par le potentiel de bruit blanc. Notre objectif est de définir l'opérateur dans l'espace $\mathbb{R}^2$ sans recourir à une théorie de renormalisation sophistiquée. Cependant, la non-bornitude du bruit blanc sur $\mathbb{R}^2$ pose des défis techniques supplémentaires. Pour surmonter cela, l'utilisation du théorème de Faris-Lavine est discutée.

MOTS CLÉS

Opérateurs de Schrödinger aléatoires, Hamiltonien d'Anderson, équations aux dérivées partielles stochastiques singulières, structures de régularité, bruit blanc, auto-adjonction.

ABSTRACT

This thesis studies the random Schrödinger operators in continuous setting, particularly those with Gaussian white noise potential. The definition of such differential operators is generally non-trivial and necessitates renormalization in dimensions $d \geq 2$. We first present a general framework to translate the problem of operator construction into stochastic PDEs. This approach enables us to define the operator at stake and establishes its self-adjointness, as well as to investigate its spectrum.
Subsequently, we proceed to study the continuous Anderson Hamiltonian under two distinct spatial settings: first on a bounded box with side length L with zero Dirichlet boundary condition for dimensions $d \leq 3$, and second on the full Euclidean space $\mathbb{R}^d$, for $d \in \{2, 3\}$. In the former case, the operator admits eigenvalues $\lambda_{n,L}$, for which we identify the almost sure asymptotic as $L \to \infty$. This asymptotic aligns with previous findings in the literature for dimension 1 and 2, while our result in dimension 3 is new. In the latter case, we propose a new construction technique employing the solution theory to the associated parabolic equation which allows to prove self-adjointness and show that the spectrum equals to $\mathbb{R}$ almost surely. This approach reconfirms the recently established result in dimension 2, but our construction seems to be more elementary; for dimension 3, our result is new.
Lastly, we present an ongoing project addressing the case where a uniform magnetic field is applied to the system: this leads to the study of Landau Hamiltonian perturbed by the white noise potential. Our objective is to define the operator on full space $\mathbb{R}^2$ without resorting to sophisticated renormalization theory. However, the unboundedness of white noise on $\mathbb{R}^2$ poses additional technical challenges. To overcome this, the usage of Faris-Lavine theorem is discussed.

KEYWORDS

Random Schrödinger operators, Anderson Hamiltonian, singular stochastic partial differential equations, regularity structures, white noise, self-adjointness.